10/15/02

Impt. sequence of #'s:

$1, \infty, 5, 6, 3, 3, 3, 3, \ldots$

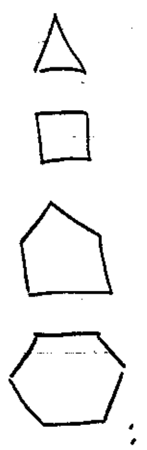
= # of regular polytopes in various dimensions.

regular polytopes in 0-d: •

$1, \; 1, \infty, 5, 6, 3, 3, 3, \ldots$

regular polytopes in 1-dim:

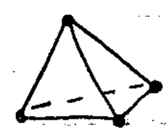
regular polygons



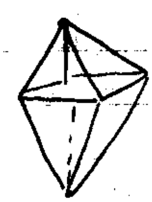
(polytopes in 2-d)
convex

regular polyhedra
(all faces are same)

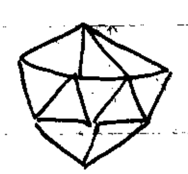
Platonic solids



tetrahedron
3 triangles per vertex

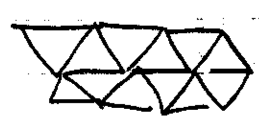


octahedron
4 triangles meet @ vertex

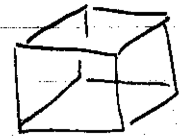


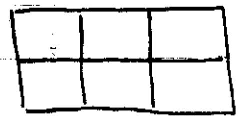
5 triangles meet @ vertex
20 sides - icosahedron

trying to do 6 triangles @ a vertex:

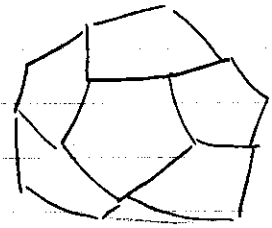


tiling of plane

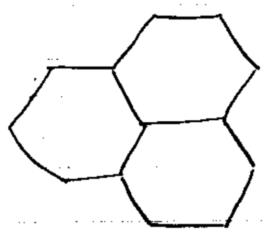
move up to squares:  cube

try w/ more squares:  tiling of plane

• move up to pentagons:



dodecahedron

can't put 3 hexagons together 

5 Platonic solids!

• Plato — relationship bet. 4 elts & these 5 solids.
hmm... predict a 5th element.

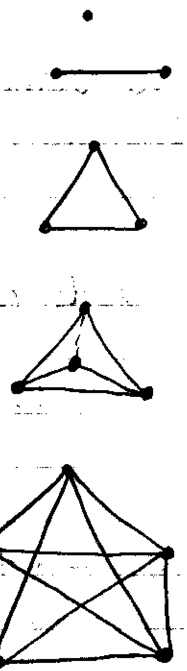
cube — the earth since you can build up things (stack well) out of cubes

icosahedron — "roundness" in a sense

continuing explaining the sequence...

The 3 infinite sequence:

n- simplices:



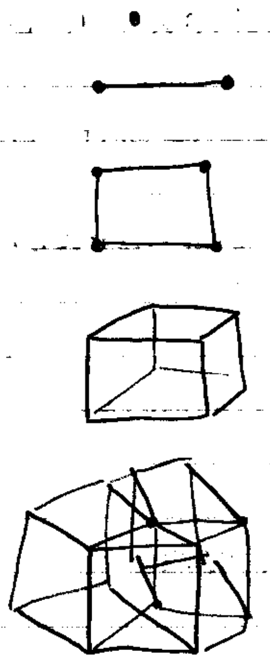
to get from  put pt in center and pull up until sides are equal.

vertices are $n+1$ points in $\mathbb{R}^n$ all equidistant from each other.

4-simplex has 5 tetrahedral faces & 5 vertices.

n-cube:

$[0,1]^n \subseteq \mathbb{R}^n$

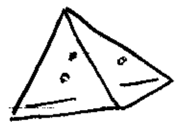


4-cube has 8 cubical faces & 16 vertices.

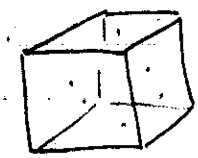
$[0,1]^4 \subseteq \mathbb{R}^4$

4th dim. move vertices from front face, back face in 4th dim

Poincaré duality:



dual is itself.



dual is octahedron
(and vice versa)

icosahedron $\xleftrightarrow{\text{dual}}$ dodecahedron

20 faces
12 vertices

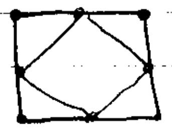
12 faces
20 vertices

Duality takes vertices $\longrightarrow$ faces
edges $\longrightarrow$ edges
faces $\longrightarrow$ vertices

* The regular polygons are all self-dual as are the regular polytopes in 0-d, 1-d.

* Every dimensional simplex is self-dual!

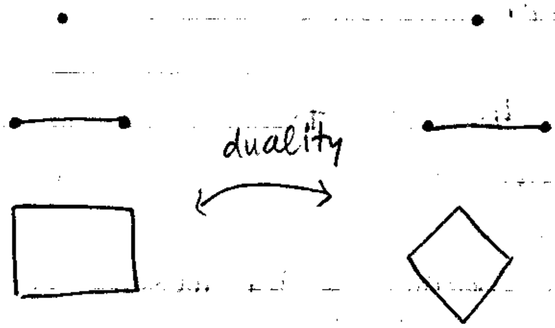
But– the cubes aren't self-dual.



(diamond)

cube $\longleftrightarrow$ octahedron (super diamond!)

vertices is $2n$.



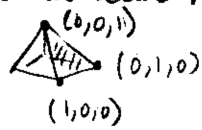
n-dim'l cross-polytopes

Vertices are:

$(\pm 1, 0)$, $(0, \pm 1)$

cube

octahedron



$(\pm 1, 0, 0)$
$(0, \pm 1, 0)$
$(0, 0, \pm 1)$

distance bet. any 2 of these pts is $\sqrt{2}$

vertices of n-dim'l cross-polytopes are $(0, \ldots, \pm 1, 0, \ldots, 0)$

The 4-dim'l cross-polytope has 16 tetrahedral faces &, 8 vertices.

Note: In dimensions 3 & 4 weird things happen in all branches of mathematics.

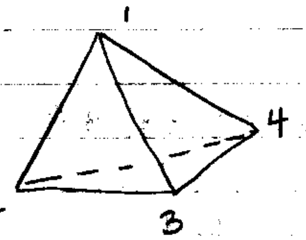
The 4-d regular polytopes:

The 4-simplex, 4-cube, 4-d cross-polytope, and 3 more!

We get these remaining 3 by looking at the symmetry groups of the Platonic solids. (So 3 & 4 dim'l stories are related.)

Jim's Dynkin diagrams seminar is related to reflections!

tetrahedron:



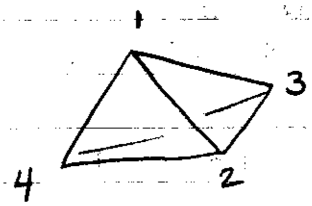
The $\boxed{\text{rotational}}$ symmetry group of any Platonic solid is a finite subgroup of $SO(3)$.

Must be a subgroup of S_4 (permutations of 4 vertices).

so can fix 1 vertex

1 2 3 4
1 4 2 3 ← cyclically permute

or



1 2 3 4
2 1 4 3

These give us only the even permutations so we get A_4

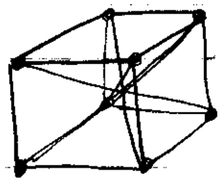
$|A_4| = 12$,

We could have figured out how many elts (12) there were:

4.3 { we have 4 choices for favorite vertex, then which of 3 faces get mapped where.

size of $|A_4| = 12$: $\underset{\text{choices of vertex}}{4} \times \underset{\text{faces per vertex}}{3}$

<u>cube</u> :



$\underset{\text{choices of vertex}}{8} \times \underset{\text{faces meet @ any vertex}}{3} = 24$ rotational symmetries.

The cube has a symmetry group of size $8 \times 3 = 24$.

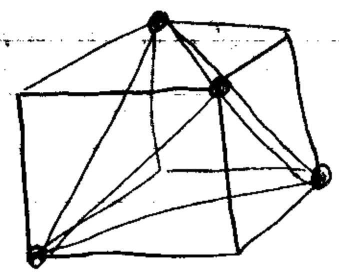
Maybe it's S_4?

green axes connect opposite corners. top – bottom diagonal

The cube has 4 diagonals and they can be permuted arbitrarily so the group is S_4.

<u>Note</u>: $A_4 \trianglelefteq S_4$ so should be relationship bet. tetrahedron & cube.

The tetrahedron is in the cube: every other vertex.



Not every perm. of S_4 preserves this green tetrahedron, but this gives

$A_4 \hookrightarrow S_4$.

vertices — faces per vertex

isocahedron: $12 \times 5 = 60$

dodecahedron: $20 \times 3 = 60$

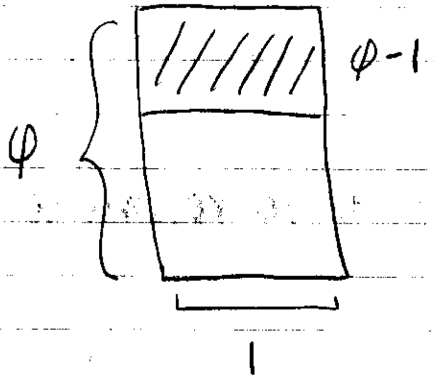
(30 edges) vertices — faces per vertex

The symmetry grp has size 60.

If we're lucky, it'll be A_5.

Is it A_5? Find 5 things in dodecahedron that get permuted.

The Greek's favorite # is the golden ratio.



φ golden ratio

golden rectangle

Euclidean algorithm shows goes on forever (must be irrational)

$\varphi : 1 :: 1 : \varphi - 1$

$$\frac{\varphi}{1} = \frac{1}{\varphi - 1}$$

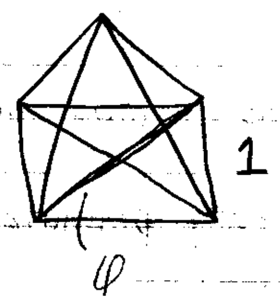
$$\frac{1}{\varphi} = \varphi - 1$$

$1 = \varphi^2 - \varphi$

$\varphi^2 - \varphi - 1 = 0$ $\qquad \varphi = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

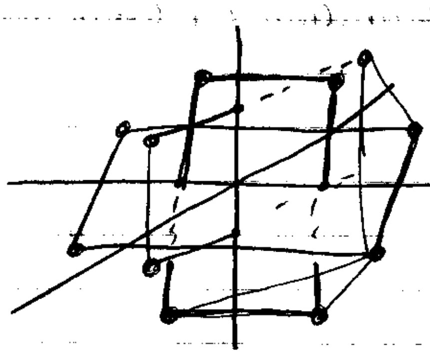
$= 1.6180339\ldots$
(post. one)

related to pentagon:



3 intersecting golden ratios

etc — connect vertices



xy plane $(\pm\varphi, \pm 1, 0)$

xz plane $(\pm 1, 0, \pm\varphi)$

yz plane $(0, \pm\varphi, \pm 1)$

this has 12 vertices these 3 intersecting planes
they're the vertices of the icosahedron

* so — golden rectangles are hiding inside icosahedron.

A <u>cross</u> is a configuration of 3 $\perp$ golden rectangles formed by the vertices of an icosahedron. There are 5 of these.

Every edge is in exactly one of these.

Each <u>cross</u> has 6 edges of an icos. in it, and icos. has 30 edges.

So— there are 5 of these crosses in the icosahedron.

* 5 ways to stick a cube in dodecahedron.

<u>Check</u>: You can get any <u>even</u> permutation of these by a symmetry.

So— symm. group of dodecahedron/icosahedron is A_5.

10/17/02 Platonic Solids (cont'd)

- we live in 4-d Minkowski spacetime
- Lie grps – grps of continuous symmetries

There are 6 regular polytopes in 4-dimensions:

1) 4-d simplex (~ tetrahedron)

2) 4-cube (~ cube)

3) 4d cross-polytope (~ octahedron)

We've talked about these; we have 3 more.
(one like icosahedron, one like dodecahedron)

We get these other 3 from symmetry groups of 3d Platonic solids:

		# elts
A_4 – tetrahedron (even perm)		12
grps	S_4 – symm. grp of cube/octahedron	24
	A_5 – symm. grp of dodecahedron/icosahedron	60

Let's start w/ A_4.

pullback square!

$$\begin{tikzcd} \tilde{A}_4 \arrow[r, hook] \arrow[d, "\substack{2\text{-}1 \\ \text{onto}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & SU(2) \arrow[d, two heads, "2\text{-}1 \text{ onto map}"] \\ A_4 \arrow[r, hook] & SO(3) \end{tikzcd}$$

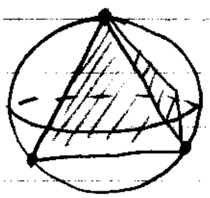
$\tilde{A}_4 \subset SU(2)$

each elt of A_4 has 2 elts from $\tilde{A}_4$ mapping to it.

$$|\tilde{A}_4| = 24$$

unit quatenions

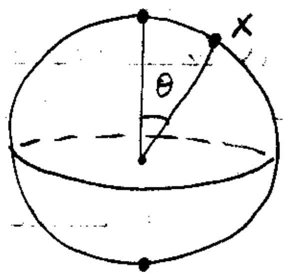
$$\begin{tikzcd}
\tilde{A}_4 \arrow[r, hook] \arrow[d] & SU(2) = S^3 \subseteq \mathbb{R}^4 \arrow[d] \\
A_4 \arrow[r, hook] & SO(3)
\end{tikzcd}$$



S^2 take convex hull

So – $\boxed{\tilde{A}_4 \text{ is the vertices of a 4d regular polytope.}}$

<u>Note</u>: Double cover of $SO(7)$ isn't a sphere. only true w/ S^3, S^2.

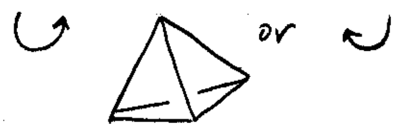


$SU(2) = S^3$

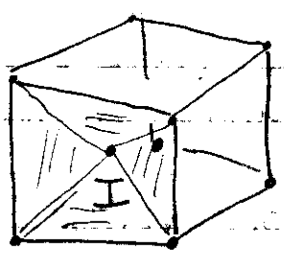
22 other pts located on this 3 sphere.
$\{1,-1\} \longmapsto id = 1$

- this elt x gets sent to rotation by 2θ in $SO(3)$.
- rotating tetrahedron 1/3 around is smallest rotation.
  So – there are 8 points in $\tilde{A}_4$ where $\theta = \pi/3$ that map to rotations by $\frac{2\pi}{3}$.

(can rotate tetrahedron 1/3 of way left or right, and can do for each of 4 vertices)



These 8 points form a cube $\subseteq S^2 \subseteq S^3$.



$\subseteq S^3$

This is a local nbhd of id. All nbhds of other pts look like this.

We've got a solid whose 2-d faces are triangles & 3-d faces are octahedra.

This polytope has 24 vertices, 96 edges

96 triangular faces

24 octahedral faces

We have 6 octahedra meet at each vertex (top, bottom, front, back, left, right)

naively you'd muet $6 \cdot \#$ of vertices but we've over counted!

$$\underbrace{24}_{\text{vertices}} \times \underbrace{6}_{\text{octahedra per vertex}} = 144 \text{ octahedra}$$

But each octahedra has 6 vertices, so we divide by 6.

Thus - it's self-dual (same # vertices & faces)
Duality - switch vertices for top dim'l faces.

<u>triangles</u>: 8 Δ's per vertex

$$\underbrace{12}_{\text{triangles per vertex}} \times \underbrace{24}_{\text{vertices}}$$

But, each triangle has 3 vertices.

$$\underbrace{24}_{\text{vertices}} \times \underbrace{12}_{\substack{\text{triangles}\\\text{per vertex}}} \div \underbrace{3}_{\substack{\#\text{ vertices}\\\text{per triangle}}} = 24 \times 4 = 96$$

$$\underbrace{24}_{\text{vertices}} \times \underbrace{8}_{\substack{\text{edges}\\\text{per vertex}}} \div \underbrace{2}_{\substack{\text{each edge has}\\\text{2 vertices}}} = 96 \text{ edges}$$

What we've described above is the 24-cell, which is self-dual and has no analogue in 3-dim.
Comes from symmetries of tetrahedron!

Symmetries of cube:

48 elts

$$\begin{tikzcd} \tilde{S}_4 \arrow[r, hook] \arrow[d] & SU(2) \subseteq S^3 \arrow[d, "\text{2-1 onto}"] \\ S_4 \arrow[r, hook] & SO(3) \end{tikzcd}$$

These are NOT the vertices of a (Platonic solid) regular polytope.

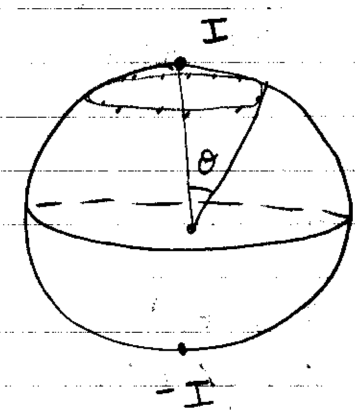
Puzzle – what solid do you get?

This new shape won't be self-dual, so we'll take it's dual.

120 elts

$$\begin{tikzcd}
\tilde{A}_5 \arrow[r, hook] \arrow[d, "\text{2-1, onto}"'] & SU(2) = S^3 \arrow[d, "\text{2-1 onto}"] \\
A_5 \arrow[r, hook] & SO(3)
\end{tikzcd}$$

$\tilde{A}_5$ form the vertices of a reg. polytope.



$SU(2) \cong S^3$

Want rotations of icosahedron have smallest angle?
It has 5-fold symmetry.
So, rotate 1/5.

\# of ways to rotate icos. by $\pi/5$.

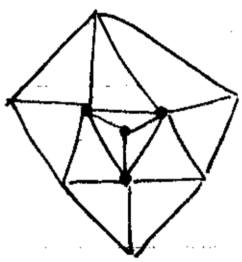
So – $\tilde{A}_5$ will have 12 vertices w/ $\theta = \pi/5$ corresponding to rotations by $2\pi/5$ that are symmetries of icosahedron.

* Here – we don't have clockwise/counterclockwise rotation!

For icosahedron – every vertex has pt. directly opposite.
Not true for tetrahedron – could rotate ↻ or ↺ since no vertex opposite of each.

These 12 pts are arranged in shape of icosahedron.

This 4d solid has 3d faces shaped like tetrahedra.



stick in each triangle a tetrahedron.

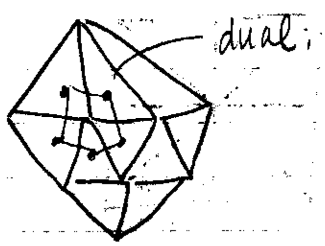
How many tetrahedra?

$$\underbrace{120}_{\text{vertices}} \times \underbrace{20}_{\text{tetrahedra per vertex}} \div \underbrace{4}_{\text{vertices per tetrahedra}} = 600$$

We call this the 600-cell.

We can take the dual of this – the 120-cell.
This is the last of the 4-d regular polytopes.
The 120-cell has 600-vertices and 120 3d faces shaped like dodecahedra.

Poincaré dual – put a vertex in each tetrahedra:



The Platonic solids also have rotation/reflection symmetry groups:

finite subgroups of $O(3)$:

# elts		
24	S_4	— we can permute vertices of tet. any way
48	$S_4 \times \mathbb{Z}_2$	
120	$A_5 \times \mathbb{Z}_2$	coming from

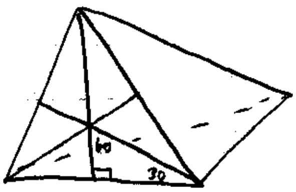
$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$ is a symm. of all but tetrahedron

(send each pt x to $-x$)

Plato had a theory of subatomic particles:

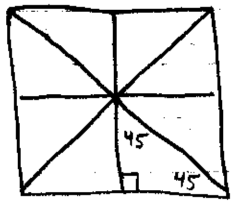
need triangular faces

barycentric subdivision of something is same as its dual.

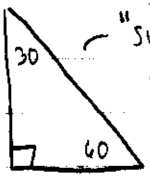


tetrahedron, octahedron, icosahedron.

But cube is different:



so



"subatomic particle"

everything made up from these!