10/22/02

<u>HW review</u>: (alternate method)

Riemann sphere $= \mathbb{C} \cup \{\infty\}$

$$\uparrow 2$$

$$\{p \in \mathcal{H} \mid p^2 = p,\ p^* = p,\ \mathrm{tr}(p) = 1\}$$

$$\downarrow 2$$

$\{$1-d subspaces of $\mathbb{C}^2\}$

Any 1-d subspace is of the form

$$\left\{\alpha\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} : \alpha \in \mathbb{C}\right\} =$$

This corresponds to the point $\left\{\alpha\begin{pmatrix}\psi_1/\psi_2\\ 1\end{pmatrix} : \alpha \in \mathbb{C}\right\}$

unless $\psi_2 = 0$.

corresponds to $z = \psi_1/\psi_2 \in \mathbb{C}$ or ∞ if $\psi_2 = 0$.

Let $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{C})$. It acts on our subspace to give

$$\left\{\alpha\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}z\\ 1\end{pmatrix} : \alpha \in \mathbb{C}\right\}$$

translate subspace back to pt. on Riemann sphere.

$$\left\{ \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} az+b \\ cz+d \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

which corresponds to $\underbrace{\frac{az+b}{cz+d}}_{= \psi_1/\psi_2} \in \mathbb{C}$

We've seen that Platonic solids give certain finite subgroups of $SO(3)$:

A_4 tetrahedron
S_4 cube
A_5 dodecahedron

normal: $gHg^{-1} = H$

"normal in math always means exceptional"

We can conjugate each of these subgroups but they just correspond to different orientations of our shapes (tetrahedron, cube, dodecahedron). So they're not really different.

Up to conjugacy the above are almost all the finite subgroups of $SO(3)$. We can also get cyclic groups $\mathbb{Z}_n$ ($n=1$ gives identity grp) and also D_n the dihedral groups ($\mathbb{Z}_n \subseteq D_n$ is of index 2; D_n is rotations/reflections of an ngon in plane).

Thm: Every finite subgroup of $SO(3)$ is conjugate to one of these:
$A_4, S_4, A_5, \mathbb{Z}_n, D_n$.

Thm: Every finite subgroup of $SO(2)$ is $\mathbb{Z}_n$.

(grp is abelian, so we don't have "conjugate")

We can easily answer this same question for $SO(4)$ using a relationship bet. $SO(4)$ é, $SO(3)$.
(almost true: $SO(4) = SO(3) \times SO(3)$ but not quite)

Let G be a finite group w/ n elts. Cook up a v. space of dim n: $\mathbb{R}^n$.
Get G to act on $\mathbb{R}^n$: "regular representation".

If G has n elts, it's a subgroup of $O(n)$.
Let G act on $\mathbb{R}[G]$ in obvious way – left mult.

(v. space w/ elts of G as basis)

$\mathbb{R}[G]$ is group ring (we just think of it as v. space)

But $O(n) \hookrightarrow SO(n+1)$ ~ (think of reflections as rotations in one dim. up)

So G is a subgrp of $SO(n+1)$.

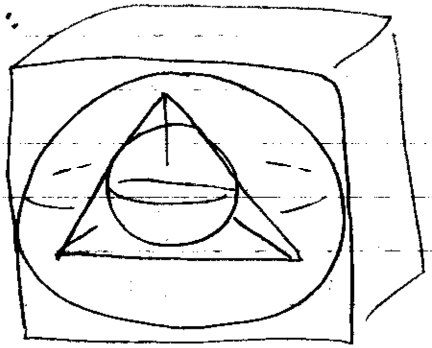
<u>Moral</u>: So – every finite group is a subgroup of <u>some</u> $SO(n)$.

We've been looking at homomorphisms <u>into</u> $SO(n)$

$$\underset{\text{finite}}{G} \hookrightarrow SO(3)$$

Plato thought these classified the elements.

Kepler nested the planets (spheres) in the Platonic solids:



Kepler thought planets moved on concentric spheres nested around Platonic solids:

6 planets: Mer
Ven
Earth
Mars
Jup
Sat

Fit radii quite well – but noticed Mars didn't go in a circle around a circle.

Next guess: ellipses! Newtonian gravity...

Shift from explaining the state of the universe to dynamical laws.

100 yrs later – studying the atom & quantum mechanics.

In quantum mechanics of atoms, we instead study homomorphisms

$$SO(3) \longrightarrow G$$

- If V is a v. space let

$$GL(V) = \{ f : V \to V \mid \text{linear, invertible} \}$$

- A homo. $\rho : G \longrightarrow GL(V)$

is called a <u>representation</u> of G.

In quantum mechanics, states are described by unit vectors in a Hilbert space, H; a homomorphism

$$\rho : G \longrightarrow U(H)$$

where $U(H) = \{ f : H \to H \mid \text{unitary} \}$ is called a <u>unitary</u> representation.

<u>Note</u> - a rep. really involves the homo & Hilbert space so it's a pair.

Given two reps (ρ, V) and (ρ', V') of G, an <u>intertwining operator</u> or <u>intertwiner</u>, from (ρ, V) to (ρ', V') is a linear map

$$f : V \longrightarrow V' \quad \text{st}$$

$f: V \longrightarrow V'$ st

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho'(g)"] \\ V \arrow[r, "f"'] & V' \end{tikzcd}$$ commutes.

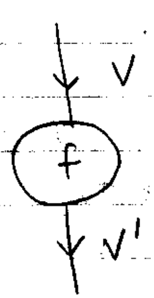
So f is a process that gets along w/ symmetry

In QM, Hilbert spaces are used to describe states of our system.
Unitary gp. reps describe how symmetries act on states

$$\underset{\overset{\uparrow}{H}}{\psi} \longmapsto \underset{\overset{\uparrow}{H}}{\rho(g)\cdot\psi}$$

Intertwiners describe processes which are compatible w/ symmetries (covariant).

We draw an intertwiner like



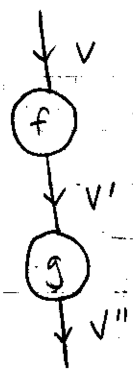
Given intertwiners

$f: (V, \rho) \longrightarrow (V', \rho')$

$g: (V', \rho') \longrightarrow (V'', \rho'')$ we get:

$gf: (V, \rho) \longrightarrow (V'', \rho'')$, an intertwiner.

We'll draw this as:



Given 2 reps (ρ, V) and (ρ', V') $\exists$ a rep

$(\rho \otimes \rho', V \otimes V')$ where $V \otimes V'$ is tensor prod. of v. spaces &

Bases
$V: e_i$
$V': e_i'$

$V \otimes V'$:
$e_i \otimes e_i'$

$(\rho \otimes \rho')(g): V \otimes V' \longrightarrow V \otimes V'$ is

$(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$ i.e.

$(\rho \otimes \rho')(g)(v \otimes v') = \rho(g)v \otimes \rho'(g)v'$

And all elts of $V \otimes V'$ are lin. combs of $v \otimes v'$, so the above determines $(\rho \otimes \rho')(g)$ since it's linear and $v \otimes v'$ span $V \otimes V'$.

Given intertwiners $f: (\rho, V) \longrightarrow (\rho', V')$

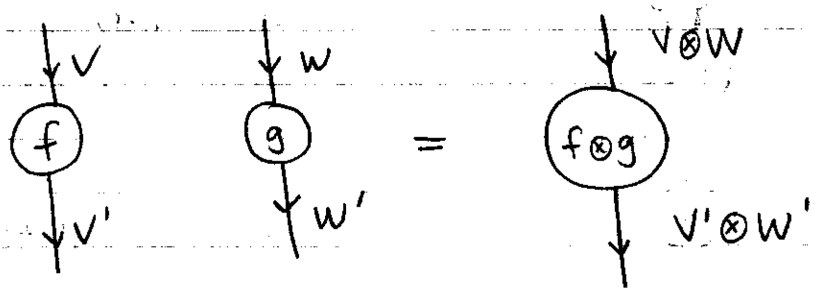
$$g: (\sigma, W) \longrightarrow (\sigma', W')$$

we get an intertwiner

$$f \otimes g: (\rho \otimes \sigma, V \otimes W) \longrightarrow (\rho' \otimes \sigma', V' \otimes W')$$

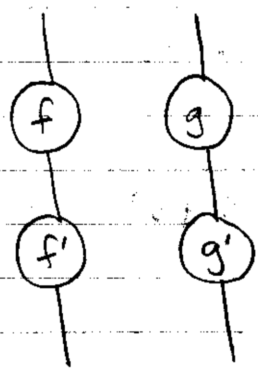
$$f \otimes g\,(v \otimes w) = f(v) \otimes g(w)$$

We draw this as:



Let the pictures do the thinking!

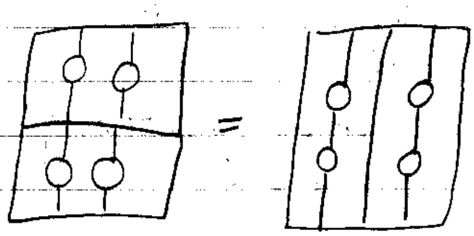
Ex)



$$f'f \otimes g'g$$
$$\parallel$$
$$(f' \otimes g')(f \otimes g)$$

can analyze this picture in 2 ways!

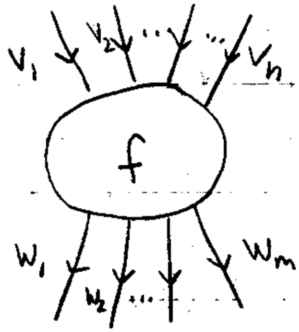
interchange law!



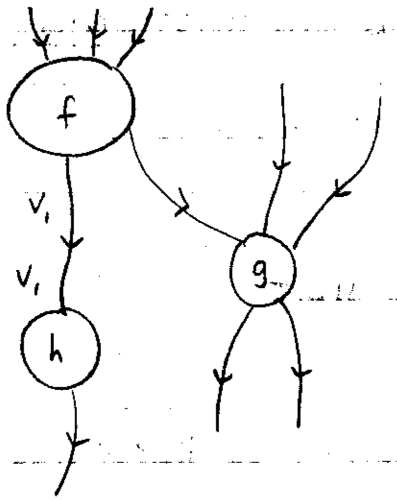
We can similarly draw intertwiners like:

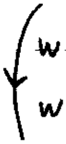
$$f: (\rho_1, V_1) \otimes \dots \otimes (\rho_n, V_n) \longrightarrow (\sigma_1, W_1) \otimes \dots \otimes (\sigma_m, W_m)$$

as follows:



We can hook these up to get intertwiners:
(using comp. & tensoring)



Recall 

is notation for $I: W \to W$ "id.

edges - called <u>particles</u>. (rep of a grp).

(f) - called an <u>interaction</u>
(vertices)

Also need to know: • the group we'll use

• what the reps of the group are

10/24/02

For any group G there is a category whose objects are representations of G:

a homo $\rho: G \longrightarrow GL(V)$

i.e. $\forall g \in G$ we get $\rho(g): V \to V$ linear st

$$\rho(gh) = \rho(g)\rho(h)$$
$$\rho(1) = 1_V$$

and whose morphisms are intertwiners

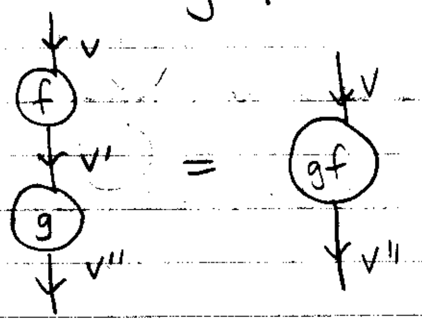
$$f: (\rho, V) \longrightarrow (\rho', V')$$

a rep which we'll call ρ for short or V for short

i.e. (defn of intertwiner) a linear map $f: V \to V'$ st.

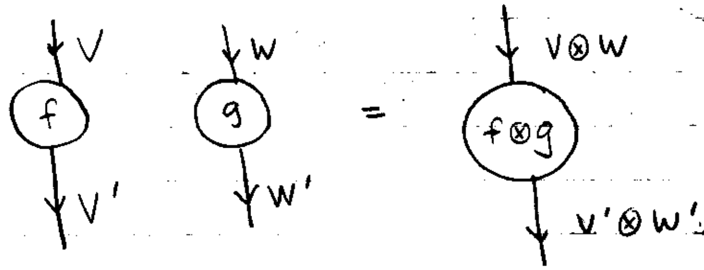
$$f\rho(g)(v) = \rho'(g)f(v) \quad \forall v \in V,\ g \in G$$

We saw this category has a 1-dim'l aspect:



$f: V \to V'$
$g: V' \longrightarrow V''$

This category is also <u>monoidal</u> meaning it has a 2-dim'l aspect to it:



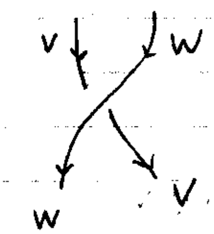
This category is also a <u>braided monoidal category</u>, meaning it has a 3-dim'l aspect as well: Given two reps V &, W we get an intertwiner, the <u>braiding</u>:

$$B_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$$
check
$$v \otimes w \longmapsto w \otimes v$$

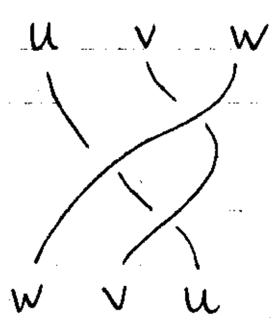
this is 1-1 &, onto &, in fact an intertwiner,

We draw the braiding as:

$B_{V,W}$:



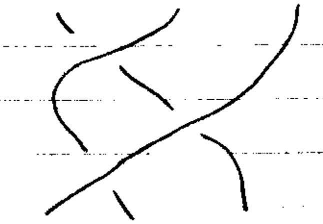
Then we get laws like:



$$(1_W \circ B_{U,V}) \circ (B_{U,W} \otimes 1_V) \circ (1_U \otimes B_{V,W})$$

3rd R. move

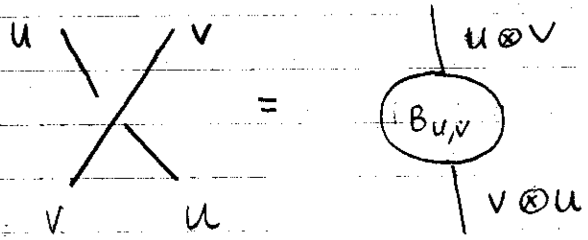
And this braiding is equal to:



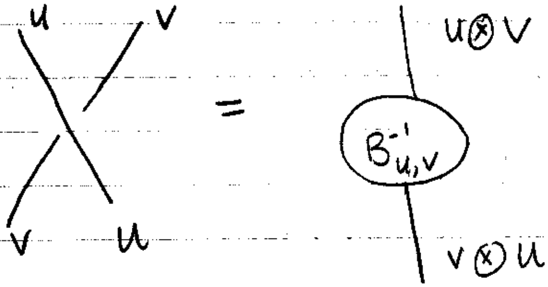
$(B_{v,w} \otimes 1_u) \cdot (1_v \otimes B_{u,w})(B_{u,v} \otimes 1_w)$

This is called <u>Yang-Baxter eqn.</u>

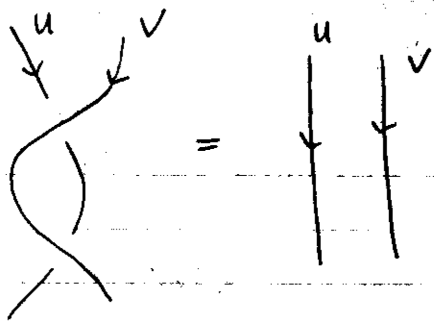
This is an isotopy! They're topologically the same picture.
Note - these crossings represent 2 particles passing &, not interacting.



Let's define:

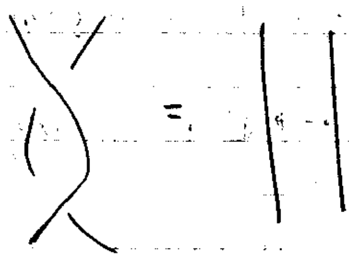


The fact that $B^{-1}_{u,v} B_{u,v} = 1_{u\otimes v}$ becomes:



$2^{\underline{nd}}$ Reidemeister move

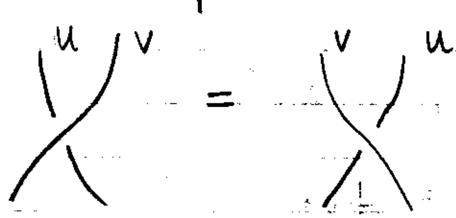
and $B_{u,v} B^{-1}_{u,v} = 1_{u\otimes v}$



There's also a 4-dim'l aspect:

In 3-dim, we can't slide things past each other — but in 4-dim we can!

So — 4-dim'l aspect is that



i.e. $B_{u,v} = B_{u,v}^{-1}$

When a braided monoidal category satisfies $B_{u,v} = B_{u,v}^{-1}$ we call it a <u>symmetric</u> <u>braided</u> <u>monoidal</u> <u>category</u>.

Note — $1 + 1 + 2 = 4$

knots (1-dim'l) — 2 for sliding around — 4d space

Group representations also have <u>duals</u> & <u>conjugates</u>.

Suppose we have a group rep (turn each grp elt into a linear transf)

$$\rho : G \longrightarrow GL(V)$$

We can get a rep on the <u>dual</u> v. space, V^*

$$V^* = \{ f \mid f : V \longrightarrow \mathbb{C} \text{ is linear} \}$$

This rep (ρ^*, V^*) is given by:

$$\boxed{(\rho^*(g)(f))(v) = f(\rho(g^{-1})v)}$$

$g \in G$
$f \in V^*$
$v \in V$

<u>Check</u>: $\rho^*(gh) = \rho^*(g)\rho^*(h)$

$$\rho^*(1_V) = 1_{V^*}.$$

Checking this involves our choice of g^{-1} above.

Recall: $\downarrow$ is ident. intertwiner.

Note: We have this linear map:

$$\varepsilon_V : V^* \otimes V \longrightarrow \mathbb{C} \qquad \text{the } \underline{\text{counit}}.$$
$$f \otimes v \longmapsto f(v)$$

$V^* \otimes V$ is a rep of G, also $\mathbb{C}$ is a rep. of G called the <u>trivial rep.</u> (group elts just act as ident.)

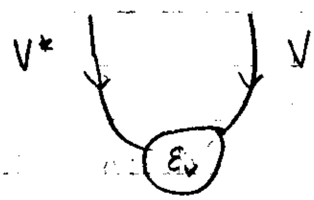
* <u>trivial rep</u>: $(\rho_{\text{trivial}}, \mathbb{C})$ has:

$$\rho_{\text{trivial}}(g) = 1_{\mathbb{C}}$$

In fact, $(\rho^*(g)(f))(v) = f(\rho(g^{-1})v)$ was cleverly chosen so that ε_V is an intertwiner.
How do we draw it?

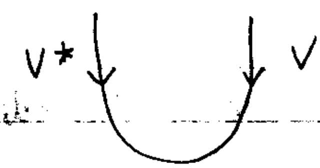
We draw $\mathbb{C}$ as nothing:

and so we draw $\varepsilon_V : V^* \otimes V \longrightarrow \mathbb{C}$ as:



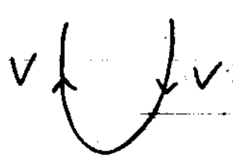
or for short:

arrow down for V, so up for V^*



or better: write $\uparrow V$ for $\downarrow V^*$

and we get



<u>antiparticle</u>.

annihilation of particle & anti-particle

Moral: If V is the space of states of some particle then V^* is the space of states of the corresponding antiparticle. (Here we're neglecting conservation of energy.)

Feynman said: "antiparticles are particles going backwards in time"

If V is finite dim'l, we also get an intertwiner called the unit

$$i_V : \mathbb{C} \longrightarrow V \otimes V^*$$

Remember: if e_i is a basis of V, then we have a dual basis of V^*, e^i, st

$$e^i(e_j) = \delta_{ij}$$

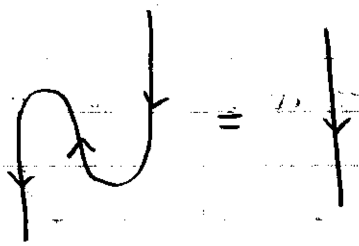
and we define:

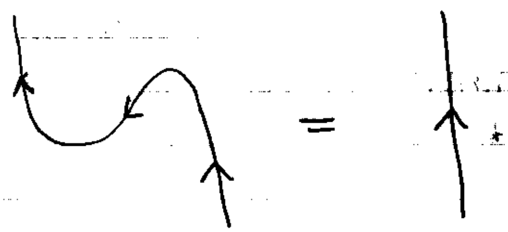
$$i_V(\alpha) = \alpha \underbrace{\sum_i e_i \otimes e^i}_{\text{doesn't depend on choice of basis}} \qquad \alpha \in \mathbb{C}$$

(secretly the ident. map $I \in \hom(V,V) \cong V \otimes V^*$)

This corresponds to "creation" of a particle-antiparticle pair.

Laws:

① 

② 

Conjugates:

We also have a conjugate representation: $(\bar{\rho}, \bar{V})$.

If we have a complex vector space V, there's a conjugate v. space $\bar{V}$ defined as follows:

As a set, $\bar{V} = V$. (As v. spaces, however, they are different.)
Given $v \in V$, to regard it as an element of $\bar{V}$ we write it as $\bar{v}$. We make $\bar{V}$ into a v. space as follows:

$$\bar{v} + \bar{w} = \overline{v+w}$$

(addition in V, then think of it as being in $\bar{V}$.)

$$\alpha \bar{v} = \overline{\bar{\alpha} v}$$

$\alpha = x + iy$

$\Rightarrow \bar{\alpha} = x - iy$

* No way to tell difference bet i & $-i$.

So – for example

$$i\bar{V} = \overline{-iv} = -\overline{iv}.$$

So we get $V, \bar{V}, V^*, \bar{V}^*, \overline{V^*}, \dots$

these 2 are naturally iso.

so we have 4 options.

$$\begin{array}{c|c} \multicolumn{2}{c}{\overset{*}{\longleftrightarrow}} \\ V^* & V \\ \hline \overline{V^*} \cong \bar{V}^* & \bar{V} \end{array}$$

– "bar" (conjugation)

- bar twice or star twice we get back where we started.

"time reversal" is related to $*$.