HW review: (alternate method)

Riemann sphere $= \mathbb{C} \cup \{\infty\}$

$\{ p \in \mathbb{H} \mid p^2 = p, \ p^* = p, \ tr(p) = 1 \}$

$\downarrow$

$\{ 1 \text{-d subspaces of } \mathbb{C}^2 \}$

Any 1-d subspace is of the form

$\{ \alpha \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) : \alpha \in \mathbb{C} \}$

This corresponds to the point $\{ \alpha \left( \begin{array}{c} 1 \\ y_2 \end{array} \right) : \alpha \in \mathbb{C} \}$ unless $y_2 = 0$.

$\downarrow$

corresponds to $z = y_1/y_2 \in \mathbb{C}$ or $\infty$ if $y_2 = 0$.

Let $g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{C})$. It acts on our subspace to give

$\{ \alpha \left( \begin{array}{c} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} 1 \\ z \end{array} \right) : \alpha \in \mathbb{C} \}$

translate subspace back to pt. on Riemann sphere.
\[ \{ \alpha \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (z) : \alpha \in \mathbb{C} \} = \{ \alpha \left( \begin{array}{c} az + b \\ cz + d \end{array} \right) : \alpha \in \mathbb{C} \} \]

which corresponds to \( \frac{az + b}{cz + d} \in \mathbb{C} \)

\[ 4, / 4_2 \]

We've seen that Platonic solids give certain finite subgroups of \( \text{SO}(3) \):

- \( A_4 \) tetrahedron
- \( S_4 \) cube
- \( A_5 \) dodecahedron

Now, we can conjugate each of these subgroups but they just correspond to different orientations of our shapes (tetrahedron, cube, dodecahedron). So they're not really different.

We can also get cyclic groups \( \mathbb{Z}_n \) (\( n=1 \) gives identity group) and also \( D_n \) the dihedral groups (\( \mathbb{Z}_n \leq D_n \) is of index 2; \( D_n \) is rotations/reflections of an \( n \)-gon in plane).

**Thm:** Every finite subgroup of \( \text{SO}(3) \) is conjugate to one of these:

- \( A_4 \), \( S_4 \), \( A_5 \), \( \mathbb{Z}_n \), \( D_n \).
Thm: Every finite subgroup of $SO(2)$ is $Z_n$.

(grp is abelian, so we don't have "conjugate")

We can easily answer this same question for $SO(4)$ using a relationship bet. $SO(4)$ & $SO(3)$.

(almost true: $SO(4)$ = $SO(3) \times SO(3)$ but not quite)

Let $G$ be a finite group w/ n elts. Cook up a

$v.$ space of dim $n$: $\mathbb{R}^n$.

Get $G$ to act on $\mathbb{R}^n$ "regular representation".

If $G$ has $n$ elts, it's a subgroup of $O(n)$.

Let $G$ act on $IR[G]$ in obvious way - left mult.

(v. space w/ elts of $G$ as basis)

But $O(n) \rightarrow SO(n+1)$ (think of reflections as rotations in one dim. up)

So $G$ is a subgroup of $SO(n+1)$.

Moral: So - every finite group is a subgroup of some $SO(n)$.

We've been looking at homomorphisms into $SO(n)$.

\[ G \rightarrow SO(3) \]

finite

Plato thought these classified the elements.
Kepler nested the planets (spheres) in the Platonic solids:

Kepler thought planets moved on concentric spheres nested around Platonic solids:

6 planets:
- Mer
- Ven
- Earth
- Mars
- Jup
- Sat

Fit radii quite well - but noticed Mars didn't go in a circle around a circle.

Next guess: ellipses! Newtonian gravity...

Shift from explaining the state of the universe to dynamical laws.

100 yrs later - studying the atom & quantum mechanics.
In quantum mechanics of atoms, we instead study homomorphisms

\[ \text{SO}(3) \rightarrow G \]

- If \( V \) is a vector space let

\[ \text{GL}(V) = \{ f : V \rightarrow V \mid \text{linear, invertible} \} \]

- A homomorphism \( \rho : G \rightarrow \text{GL}(V) \)

is called a representation of \( G \).

In quantum mechanics, states are described by unit vectors in a Hilbert space, \( \mathcal{H} \); a homomorphism

\[ \rho : G \rightarrow U(\mathcal{H}) \]

where \( U(\mathcal{H}) = \{ f : \mathcal{H} \rightarrow \mathcal{H} \mid \text{unitary} \} \)

is called a unitary representation.

Note - a rep. really involves the hom \( G \), Hilbert space so it's a pair.

Given two reps \((\rho, V)\) and \((\rho', V')\) of \( G \), an intertwining operator or intertwiner is a linear map

\[ f : V \rightarrow V' \quad \text{st} \]

\[ \rho(g)(v) = \rho'(g)(f(v)) \]

for all \( g \in G \) and \( v \in V \).
\[ f: V \rightarrow V' \]  

\[ V \xrightarrow{f} V' \xrightarrow{\rho'(g)} \]  

\[ V \xrightarrow{f} V' \]  

Commutes.

So, \( f \) is a process that gets along w/ symmetry.

In QM, Hilbert spaces are used to describe states of our system. Unitary group reps describe how symmetries act on states:

\[ H \xrightarrow{\rho(g)} H \]

Intertwiners describe processes which are compatible w/ symmetries (covariant).

We draw an intertwiner like:

\[ V \xrightarrow{f} V' \]
Given intertwiners

\[ f: (V, \rho) \rightarrow (V', \rho') \]

\[ g: (V', \rho') \rightarrow (V'', \rho'') \]

we get:

\[ gf: (V, \rho) \rightarrow (V'', \rho''), \text{ an intertwiner.} \]

We'll draw this as:

```
\[
\begin{array}{c}
V \\
\downarrow f \\
V' \\
\downarrow g \\
V''
\end{array}
\]
```

Given 2 reps \((\rho, V)\) and \((\rho', V')\) there is a rep

\[(\rho \otimes \rho', V \otimes V')\]

where \(V \otimes V'\) is tensor product of \(V\) spaces \(\xi_i\),

\[ (\rho \otimes \rho')(g): V \otimes V' \rightarrow V \otimes V' \]

is

\[ (\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g) \]

i.e.

\[ (\rho \otimes \rho')(g)(V \otimes V') = \rho(g)V \otimes \rho'(g)V' \]

And all elts of \(V \otimes V'\) are lin. combs of \(V \otimes V'\), so the above determines \((\rho \otimes \rho')(g)\) since it's linear and \(V \otimes V'\) span \(V \otimes V'\).
Given intertwiners \( f : (\rho, \nu) \to (\rho', \nu') \)

\( g : (\sigma, \omega) \to (\sigma', \omega') \)

we get an intertwiner

\[ f \odot g : (\rho \odot \sigma, \nu \odot \omega) \to (\rho' \odot \sigma', \nu' \odot \omega') \]

\[ f \odot g (\nu \odot \omega) = f(\nu) \odot g(\omega) \]

We draw this as:

\[ f \]

\[ \downarrow \nu' \]

\[ g \]

\[ \downarrow \omega' \]

\[ f \odot g \]

\[ \downarrow \nu' \odot \omega' \]

Let the pictures do the thinking!

EX):

\[ f' \odot g' \]

\[ f \odot g \]

\[ (f' \odot g')(f \odot g) \]

can analyze this picture in 2 ways!

interchange law!
We can similarly draw intertwiners like:

\[ f: (\rho_i, v_i) \otimes \cdots \otimes (\rho_n, v_n) \rightarrow (\sigma_i, w_i) \otimes \cdots \otimes (\sigma_m, w_m) \]

as follows:

We can hook these up to get intertwiners:

(using comp. & tensoring)

Recall \( f^w \) is notation for \( I: w \rightarrow w \)

"id.

edges - called \underline{particles}. (rep of a grp)

\( f \) - called an \underline{interaction} (vertices)

Also need to know: the group we'll use.

what the reps of the group are
For any group $G$, there is a category whose objects are representations of $G$:

A homomorphism $\rho: G \to GL(V)$

i.e. $\forall g \in G$ we get $\rho(g): V \to V$ linear such that $\rho(gh) = \rho(g)\rho(h)$

$\rho(1) = 1_V$

and whose morphisms are *intertwiners* $f: (\rho, V) \to (\rho', V')$

which we'll call $\rho$ for short or $V$ for short.

i.e. (defn of intertwiner) a linear map $f: V \to V'$ such that $f\rho(g)(v) = \rho'(g)f(v)$ $\forall v \in V, g \in G$

We saw this category has a 1-dim'l aspect:

\[
\begin{aligned}
&f & &\leq & &\leq & &f: V \to V' \\
&g & &\leq & &\leq & &g: V' \to V'' \\
&\circ & &\leq & &\leq & &gf \\
\end{aligned}
\]
This category is also monoidal meaning it has a 2-dim'el aspect to it:

\[
\begin{array}{ccc}
  V & \xrightarrow{f} & V' \\
  \downarrow & & \downarrow \\
  w & \xrightarrow{g} & w'
\end{array}
\]

\[
\begin{array}{ccc}
  v & \xrightarrow{\text{fog}} & v'
\end{array}
\]

This category is also a braided monoidal category, meaning it has a 3-dim'el aspect as well:
Given two reps \( V, W \) we get an intertwiner, the braiding:

\[
B_{V,W}: V \otimes W \xrightarrow{\sim} W \otimes V
\]

\[
\begin{array}{ccc}
  v \otimes w & \xrightarrow{B_{V,W}} & w \otimes v
\end{array}
\]

this is 1-1 \& onto \& in fact an intertwiner,
we draw the braiding as:

\[
B_{V,W}: V \leftrightarrow W
\]

Then we get laws like:

\[
(1_w \circ B_{u,v}) \circ (B_{u,w} \otimes 1_v) = (1_u \otimes B_{v,w})
\]
And this braiding is equal to:

\[ (B_{v,w} \otimes 1_w) \cdot (1_v \otimes B_{u,w}) \cdot (B_{u,v} \otimes 1_v) \]

This is called the Yang-Baxter eqn.

This is an isotopy! They're topologically the same picture.

Note - these crossings represent 2 particles passing each other, not interacting.

Let's define:

\[ (u \otimes v) = B_{u,v} \cdot (v \otimes u) \]

\[ (u \otimes v) = B_{u,v}^{-1} \cdot (v \otimes u) \]
The fact that $B_{u,v}^{-1} \cdot B_{u,v} = 1_{u \otimes v}$ becomes:

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\]

2nd Reidemeister move

and $B_{u,v} \cdot B_{u,v}^{-1} = 1_{u \otimes v}$

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\]

There's also a 4-dim'le aspect:

In 3-dim, we can't slide things past each other —
but in 4-dim we can!

So — 4-dim'le aspect is that

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\end{array}
\]

i.e. $B_{u,v} = B_{u,v}^{-1}$

When a braided monoidal category satisfies $B_{u,v} = B_{u,v}^{-1}$
we call it a symmetric braided monoidal category.
Note: \( 1 + 1 + 2 = 4 \)

\[ g \mapsto \begin{array}{c}
\rightarrow \quad \text{4d space}
\end{array} \]

Knots 2 for (1-dim'l) sliding around

Group representations also have duals \( \rho^* \) conjugates.

Suppose we have a group rep (turn each group into a linear transf)

\[ \rho: G \rightarrow GL(V) \]

We can get a rep on the dual \( V^* \)

\[ V^* = \{ f \mid f: V \rightarrow \mathbb{C} \text{ is linear} \} \]

This rep \( (\rho^*, V^*) \) is given by:

\[ (\rho^*(g)(f))(v) = f(\rho(g^{-1})v) \]

\( g \in G \)
\( f \in V^* \)
\( v \in V \)

Check: \( \rho^*(gh) = \rho^*(g)\rho^*(h) \)

\[ \rho^*(1_v) = 1_{V^*} \]

Checking this involves our choice of \( g^{-1} \) above.
We have this linear map:
\[
\varepsilon_v : V^* \otimes V \rightarrow \mathbb{C} \quad \text{the counit.}
\]
\[
f \circ v \quad \quad \rightarrow \quad f(v)
\]

\(V^* \otimes V\) is a rep of \(G\), also \(\mathbb{C}\) is a rep. of \(G\) called the \textbf{trivial rep.} (group elts just act as ident.)

\textbf{trivial rep.}: \((\rho_{\text{trivial}}, \mathbb{C})\) has:
\[
\rho_{\text{trivial}}(g) = \frac{1}{c}
\]

In fact, \((\rho^+(g)(f))(v) = f(\rho(g^{-1})v)\)

was cleverly chosen so that \(\varepsilon_v\) is an \textit{intertwiner}.
How do we draw it?

We draw \(\mathbb{C}\) as nothing:

and so we draw \(\varepsilon_v : V^* \otimes V \rightarrow \mathbb{C}\) as:

\[
\begin{array}{c}
V^* \\
\varepsilon_v \\
\end{array}
\begin{array}{c}
\downarrow
V
\end{array}
\]
or for short:

\[
\begin{array}{c}
V^* \\
\varepsilon_v \\
V
\end{array}
\]

arrow down for \(V\), so up for \(V^*\)

or better: write \(\uparrow v\) for \(\downarrow v^*\)

and we get

\[
\begin{array}{c}
v
\end{array}
\begin{array}{c}
\uparrow v \\
\downarrow v^* \\
\text{annihilation of particle}, \text{antiparticle}
\end{array}
\]
Moral: If $V$ is the space of states of some particle then $V^*$ is the space of states of the corresponding antiparticle. (Here we're neglecting conservation of energy.)

Feynman said: "antiparticles are particles going backwards in time".

If $V$ is finite dimensional, we also get an intertwiner called the unit

$$i_v : \mathbb{C} \rightarrow V \otimes V^*$$

Remember: if $e_i$ is a basis of $V$, then we have a dual basis of $V^*$, $e_i^*$, such that $e_i^*(e_j) = \delta_{ij}$.

And we define:

$$i_v (\alpha) = \alpha \sum_i e_i \otimes e_i^* \quad \alpha \in \mathbb{C}$$

(doesn't depend on choice of basis)

(secretly the identity map $I \in \text{Hom}(V, V) = V \otimes V^*)$

This corresponds to "creation" of a particle-antiparticle pair.

$$\overset{\text{creation}}{V} \rightarrow V$$
Laws:

1. \[ \frac{1}{2} = \frac{1}{2} \]

2. \[ \frac{1}{2} = \frac{1}{2} \]

Conjugates:

We also have a conjugate representation: \((\bar{\vec{p}}, \bar{\vec{v}})\).

If we have a complex vector space \(V\), there's a conjugate \(V\) space \(\bar{V}\) defined as follows:

As a set, \(\bar{V} = V\). (As \(V\) spaces, however, they are different.)

Given \(v \in V\), to regard it as an element of \(\bar{V}\) we write it as \(\bar{v}\). We make \(\bar{V}\) into a \(V\) space as follows:

\[ \bar{v} + \bar{w} = \bar{v + w} \quad \text{(addition in } V\text{, then think of it as being in } \bar{V}) \]

\[ \alpha \bar{v} = \bar{\alpha v} \]

\[ \alpha = x + iy \]

\[ \Rightarrow \bar{\alpha} = x - iy \]
So, for example

\[ i\bar{V} = -iv = -\bar{iv} \]

So we get \( V, \bar{V}, V^*, \bar{V}^* \), \( \ldots \)

these 2 are naturally iso.

so we have 4 options.

\[ \begin{array}{c|c}
V^* & \bar{V}^* \\
\hline
V & \bar{V} \end{array} \]

\[ \sum \text{ "bar" (conjugation)} \]

... bar twice or star twice we get back where we started.

"time reversal" is related to *.