10/29/02

In quantum field theory there are 3 famous discrete symmetries a theory might have:

- C — charge conjugation (replacing particles by antiparticles)
  $C^2 = I$
  (not a 1-para. grp)

- P — parity (invert space: $(x, y, z) \longmapsto (-x, -y, -z)$)
  $P^2 = I$

- T — time reversal (reflect time: $t \longmapsto -t$)
  $T^2 = I$

CPT Thm: CPT is a symmetry of any reasonable QFT.

But: P is not, C is not, T is not, CP is not...
only the combination of the three is.

We see glimmerings of these symmetries in group representation theory.

Suppose (ρ, V) is a rep of a group G.

(homo) $\rho: G \longrightarrow GL(V)$

What are (ρ^*, V^*) (dual rep) and $(\bar{\rho}, \bar{V})$ (conj. rep) like?

Let V be a v. space. Then

- $V^* = \{ f: V \longrightarrow \mathbb{C} \mid f \text{ linear} \}$

$\overline{V} = V$ made into a v. space st $\alpha\overline{v} = \overline{(\bar{\alpha}v)}$
w/ $v \in V$ corresponding to $\overline{v} \in \overline{V}$.

Given a linear map $T: V \longrightarrow W$ we get

<u>adjoint of T</u>: $T^*: W^* \longrightarrow V^*$ (akin to time reversal)

<u>conjugate of T</u>: $\overline{T}: \overline{V} \longrightarrow \overline{W}$ as follows:

($*$ and $-$ are things you can do to v. spaces or the linear maps bet. them. They're functors!)

Let's write

$$(f, v) = f(v) \qquad v \in V,\ f \in V^*$$

and similarly for W. Then T^* is defined by:

$$\boxed{(T^*f, v) = (f, Tv)}$$

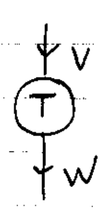
$(T^*f)(v)$ $\qquad\qquad$ $f(Tv)$

for $v \in V$
$f \in W^*$
$Tv \in W$
$T^*f \in V^*$

Note – this isn't an inner product

Note – if $T: V \longrightarrow W$, $S: W \longrightarrow X$ then

$$(ST)^* = T^* S^* \qquad \text{(contravariant)}$$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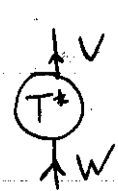
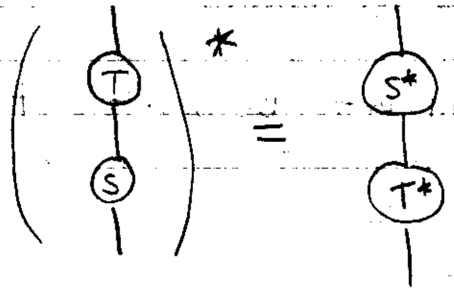


(up arrows mean dual $*$)



$$((ST)^* f, v) = (f, STv)$$
$$= (S^* f, Tv)$$
$$= (T^* S^* f, v)$$

Let's now define $\overline{\overline{T}} : \overline{V} \longrightarrow \overline{W}$.

$$\overline{\overline{T}}\,\overline{v} = \overline{Tv} \qquad \text{(covariant)}$$

<u>note</u>: $\overline{(ST)} = \overline{S}\,\overline{\overline{T}}$

<u>pf</u>: $\overline{(ST)}(\overline{v}) = \overline{S(Tv)}$

by defn of $\overline{S}$

$= \overline{S}(\overline{Tv})$

defn of $\overline{\overline{T}}$.

$= \overline{S}\,\overline{\overline{T}}\,\overline{v}$

Check that $\overline{\overline{T}}$ is linear: Yes!

$$\overline{\overline{T}}(\alpha\overline{v}) \overset{?}{=} \alpha\,\overline{\overline{T}}\,\overline{v} \quad \text{for } v \in V.$$

$$\overline{\overline{T}}(\alpha\overline{v}) = \overline{\overline{T}}(\overline{\overline{\alpha} v}) = \overline{T(\overline{\alpha} v)} = \overline{\overline{\alpha}\, T(v)}$$

defn of $\alpha\overline{v}$

T is linear

defn scalar mult

$= \alpha\,\overline{T(v)}$

defn $\overline{\overline{T}}$

$= \alpha\,\overline{\overline{T}}\,\overline{v}$

Note: BUT $T \longmapsto \overline{T}$ is not linear.

In physics, our reps will often be on Hilbert spaces. If we have a rep. ρ on a Hilbert space H we say it is unitary if

$$\rho(g): H \longrightarrow H \quad \text{is unitary } \forall g \in H.$$

unitary - means we preserve the inner product.

i.e. $\rho: G \longrightarrow U(H)$

$$U(H) = \left\{ f: H \to H \;\middle|\; \begin{array}{l} f \text{ is linear, 1-1, onto} \\ \text{and preserves inner products:} \\ \langle fv, fw \rangle = \langle v, w \rangle \\ \forall v, w \in H \end{array} \right\}$$

$\boxed{\text{unitary: 1-1, onto \& linear \& } \langle fv, fw \rangle = \langle v, w \rangle \ \forall v, w \in H.}$

Thm: If (ρ, H) is a unitary rep of G then the rep (ρ^*, H^*) is unitarily equivalent to $(\overline{\rho}, \overline{H})$.

Defn: Recall — given 2 reps (ρ, V) and (ρ', V') we say that they're equivalent if there's an intertwiner
$f: V \longrightarrow V'$ which has an inverse
$f^{-1}: V' \longrightarrow V$ (automatically an intertwiner).

Or — f is 1-1 & onto.

Inner product on $\overline{H}$: $\langle \bar{v}, \bar{w} \rangle = \overline{\langle v, w \rangle} = \langle w, v \rangle$.

To get inner product on H^*, we'll use our equivalence to transfer $\langle \cdot , \cdot \rangle$ from $\overline{H}$ to H^*.

Given 2 unitary reps (ρ, H) and (ρ', H') we say: they're <u>unitarily equivalent</u> if <u>they're equivalent</u> by an intertwiner

$f: H \longrightarrow H'$ which is <u>unitary</u>.

$$\underbrace{\qquad}_{\text{linear, 1-1, onto, \& } \langle fv, fw \rangle = \langle v, w \rangle}$$

Ex) $(\alpha_1, \ldots,) \in \ell^2$

$\downarrow$

$(0, \alpha_1, \ldots) \in \ell^2 = H$ $\quad f: H \longrightarrow H$

lin., 1-1, $\langle fv, fw \rangle = \langle v, w \rangle$

but not onto.

<u>proof of Thm on prev pg:</u>

We need a 1-1, onto intertwiner $f: \overline{H} \longrightarrow H^*$.

Note – a linear map $f: \overline{V} \longrightarrow W$ is the same (1-1 corresp.) as an antilinear (i.e. conjugate-linear) map

$$\tilde{f}: V \longrightarrow W \quad \text{st}$$

antilinear condition
$$\begin{cases} \tilde{f}(v+w) = \tilde{f}(v) + \tilde{f}(w) \\ \tilde{f}(\alpha v) = \bar{\alpha} \tilde{f}(v) \end{cases}$$

Thought of as a map $f: \overline{V} \longrightarrow W$, it's <u>linear</u>:

$$f(\alpha \bar{v}) = f(\overline{\bar{\alpha} v}) = \tilde{f}(\bar{\alpha} v) = \bar{\bar{\alpha}} \tilde{f}(v) = \alpha \tilde{f}(v)$$

$$= \alpha f(\bar{v})$$

Note: $f(\overline{\Box}) = \tilde{f}(\Box)$.

So we need antilinear $\tilde{f}: H \longrightarrow H^*$ and here it is:

$$\boxed{v \longmapsto \langle v, \cdot \rangle}$$

Hilbert spaces iso to their duals, but maps aren't linear - they're conj. linear

<u>Note</u>: physics convention $\langle \cdot , \cdot \rangle$

antilinear in 1st slot (conjugate-linear)

linear

This map <u>is</u> <u>antilinear</u>:

$$\langle \alpha v, \cdot \rangle = \bar{\alpha} \langle v, \cdot \rangle .$$

rest of proof — check this is an intertwiner:

$$f \bar{\rho}(g) = \rho^*(g) f .$$

and — it's 1-1 & onto. ☒

<u>Note</u>: $*$ is PT, $-$ is C, so prev. thm is CPT thm!

Now — let's examine the <u>reps</u> & <u>unitary</u> <u>reps</u> of various groups including:
(want Lie grps)

- $(\mathbb{R}, +)$ this is time translation symmetry (could also be translation in x, y, z dir)

- $(S^1, \cdot) = (U(1), \cdot)$ phase rotations (in electromagnetism)

  $\{z \mid |z| = 1\}$ "phases"

- if time were a circle — then this would be time translation. (statistical physics)

2x2 Herm. matrices

in fact all grps up here are the universal covers

$$
\begin{tikzcd}
SU(2) \arrow[r, hook] \arrow[d, two heads, "\text{2-1, onto}"] & SL(2,\mathbb{C}) \arrow[r, hook] \arrow[d, two heads, "\text{2-1, onto}"] & SL(2,\mathbb{C}) \ltimes \mathcal{H} = \tilde{P} \arrow[d, two heads, "\text{2-1 onto}"] \\
SO(3) \arrow[r, hook] & SO_0(3,1) \arrow[r, hook] & \underbrace{SO_0(3,1) \ltimes \mathbb{R}^4}_{\text{Poincaré grp. (10 dim'l)}} = P
\end{tikzcd}
$$

$SO(3)$: rotations in 3-dim'l space

$SO_0(3,1)$: connect. comp.

elts in $SO_0(3,1)$ are 4x4 matrices. Can act on $\mathbb{R}^4$

$$(L, v)(L', v') = (LL', v + Lv')$$

$SO_0(3,1)$ acts as an auto of $\mathbb{R}^4$

* $\tilde{P}$ is the double cover, in fact <u>universal cover</u> of Poincare' grp P.

$\tilde{P}$ is the symmetry group in QFT on 4d Minkowski spacetime.

* All these groups are <u>Lie groups</u>:

groups that are smooth manifolds w/ group operations

$$m: G \times G \longrightarrow G \qquad (g,h) \longmapsto gh$$
$$inv: G \longrightarrow G \qquad g \longmapsto g^{-1}$$

being <u>smooth</u>.

In this context we always demand that our reps

$$\rho: G \longrightarrow GL(V) \text{ be smooth.}$$

Then we can differentiate them:

$$d\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

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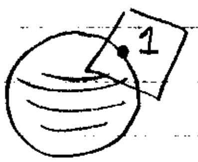
We're mainly interested in reps of Lie groups:

$$\rho : G \longrightarrow GL(V).$$

(we like Lie grps because then we can do Calculus!)
We'll assume ρ is smooth.

If V is finite dim'l, then as soon as ρ is continuous, it's smooth. In fact, ρ is smooth as soon as ρ is <u>measurable</u>. We can locally think of ρ in terms of n measurable functions (think of $GL(V)$ as $\mathbb{R}^n$).

Then we can differentiate it



$$T_e G \cong \mathfrak{g}$$

- can almost know everything about the Lie grp knowing about the Lie alg.
- bracket corresponds to mult in grp.

Differentiating, we get:

$$d\rho : \underset{\substack{\| \\ T_e G}}{\mathfrak{g}} \longrightarrow \underset{\substack{\| \\ T_e GL(V)}}{\mathfrak{gl}(V)}$$

where $\mathfrak{g}$ and

$$\mathfrak{gl}(V) = \{ f : V \longrightarrow V \mid f \text{ is linear} \}$$ are Lie algs

where $[\cdot,\cdot]$ in $\mathfrak{gl}(V)$ is just $[x,y] = xy - yx$.

and similarly for $\mathfrak{g}$ if G is a matrix Lie group
e.g.
$SO(3)$, $SU(2)$, $SL(2,\mathbb{C})$.

<u>representation</u>: is a homo <u>into</u> a really nice space — like a grp of matrices.

$d\rho$ will be a <u>Lie algebra</u> <u>rep</u>., i.e. a Lie alg homomorphism

$$d\rho(x+y) = d\rho(x) + d\rho(y)$$

$$d\rho(\alpha x) = \alpha\, d\rho(x)$$

$$d\rho([x,y]) = [d\rho(x), d\rho(y)]$$

<u>Note</u>: $d\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a linear map bet v.spaces,
so it's easier to study this Lie alg rep than a Lie grp rep.

<u>True</u>: We can always take a Lie grp homo and differentiate to get a Lie alg homo (rep).

But we can't always go backwards...

We can almost go back and get a rep

$$\int \rho : G \longrightarrow GL(V)$$

from any Lie alg rep:

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

* We can do this if G is simply connected & connected.

true in infinite dim'l case?

Thm: There is a 1-1 correspondence bet. Lie grp reps of G and Lie alg. reps of $\mathfrak{g}$ if G is connected & simply connected.

Thm: For any Lie algebra $\mathfrak{g}$ there is a connected and simply connected G w/ $\mathfrak{g}$ as its Lie algebra and G is unique (up to iso).

Example: $\mathfrak{g} = \mathfrak{su}(2) = \{ T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \mid T \text{ linear } \& \ tr = 0, \ T^* = -T \}$

antihermitian

$= \{ i(x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3) \mid x_i \in \mathbb{R} \}$ 3 dim'l Lie alg

$= \left\{ \begin{pmatrix} ix_1 & -i(x_2 + ix_3) \\ i(x_2 + ix_3) & -ix_1 \end{pmatrix} \right\}$

$[\cdot,\cdot]$ in here is the same as cross product on $\mathbb{R}^3$ under isomorphism:

$$\mathbb{R}^3 \longmapsto \mathfrak{su}(2)$$

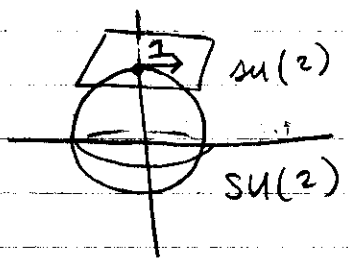
$$(x_1, x_2, x_3) \longmapsto \begin{pmatrix} ix_1 & -i(x_2 + ix_3) \\ i(x_2 + ix_3) & -ix_1 \end{pmatrix}$$

So – $su(2)$ w/ $[\cdot,\cdot]$ is $\mathbb{R}^3$ w/ $\times$
'cross product'.

SU(2) has $su(2)$ as its Lie algebra.

$$SU(2) = \left\{ T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \;\middle|\; \begin{array}{l} T \text{ linear}, \det(T) = 1 \\ TT^* = 1 \end{array} \right\}$$

* Differentiating conditions $\det(T) = 1$ and $TT^* = 1$
we get
$$tr(T) = 0 \;\; \& \;\; T^* = -T.$$



SU(2) is simply connected. But coffee cup trick tells us SO(3) isn't. And SU(2) is double cover of SU(2).

SO(3) also has $su(2)$ as its Lie alg. (well, something iso to $su(2)$).
But SO(3) isn't simply connected, and SU(2) is.

$$\begin{array}{ccc} SU(2) & & \pi_1(SU(2)) \cong e \\ \big\downarrow{\scriptstyle 2\text{-}1} & & \\ SO(3) & & \pi_1(SO(3)) \cong \mathbb{Z}_2 \end{array}$$

2 ways to rotate.

$SU(2)$ is the <u>universal cover</u> of $SO(3)$. i.e. there's a homo

$$\rho: SU(2) \xrightarrow{\text{onto}} SO(3) \quad (\text{locally } 1\text{-}1)$$

and

$SU(2)$ is simply connected.

There's a 1-1 correspondence bet reps of $SU(2)$ and $su(2)$, but not bet. $SO(3)$ and $su(2)$.

Let's look at representations of various Lie grps & Lie algs.

<u>Example 1</u>: $G = (\mathbb{R}, +, 0)$

iden. elt

operation

A rep of G is a (smooth) map

$$\rho: \mathbb{R} \longrightarrow GL(V)$$

$$\rho(t+s) = \rho(t)\rho(s) \quad \text{for all } t, s \in \mathbb{R}$$
$$\rho(0) = 1_V$$

on RHS $\rho(t)\rho(s)$ is matrix mult (comp of lin maps)

<u>Notice</u>: the exponential satisfies these conditions!

How do we think of a real # as a linear transf on a 1-dim'l v. space? Just a 1×1 matrix that is scalar mult.

So - if we take $V = \mathbb{C}$, then

$GL(V) = \{$invertible 1×1 complex matrices$\}$

$= \{$invertible complex #'s$\}$

Then one rep is.

$$\boxed{\rho(t) = e^t \quad \forall\ t \in \mathbb{R}}$$

changes addition to mult.

* This is why the exponential map is good! It's a rep!

• Also, if $V = \mathbb{C}^2$, then

$GL(V) = \{$invertible 2×2 matrices$\}$

Let $\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then check:

$\rho(t+s) = \rho(t)\rho(s)$ express add in terms of matrix mult.

$$\begin{pmatrix} 1 & t+s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

- trivial rep: when $\rho = 1$
- We can alter $\rho(t) = e^t$ by a constant:

In general we get lots of 1-dim'l reps by letting $\boxed{\rho_k(t) = e^{kt} \text{ for any } k \in \mathbb{C}.}$

Note: We don't get anything new by using 'e' as our base:

e.g. $2^{kt} = e^{\ln 2 kt}$ just change k to get back.

$\boxed{\text{Thm:}}$ Every 1-dim'l (smooth) rep of $(\mathbb{R}, +, 0)$ is equivalent to a rep of the form ρ_k for some $k \in \mathbb{C}$.

Furthermore — ρ_K is equiv. to $\rho_{K'}$ iff $K = K'$.

Here's a non-smooth, 1-dim'l rep. of $\mathbb{R}$:

Think of $\mathbb{R}$ as a v. space over $\mathbb{Q}$. (i.e. a free module over $\mathbb{Q}$)

$$\mathbb{R} \cong \mathbb{Q}^X \text{ — an uncountable}$$

where X labels a 'Hamel basis' of $\mathbb{R}$:

given $\alpha \in X$, get $X_\alpha \in \mathbb{R}$ s.t. $\forall\ t \in \mathbb{R}$ we have

$$t = \sum_{\alpha \in X} q_\alpha X_\alpha \quad q_\alpha \in \mathbb{Q}.$$

Now let

$$\rho(t) = e^{g\alpha}.$$

Then—

$$\rho(s+t) = \rho(s)\rho(t)$$
$$\rho(0) = 1.$$

This is a non-measurable function.

Note: (the graphs of: non-linear, additive functs are dense in the plane)

We want to understand all reps of $\mathbb{R}$:
How do we get more reps of a group given some?

Given (ρ, V), we get:

(ρ^*, V^*) dual rep

$(\bar{\rho}, \bar{V})$ conj. rep.

Example: Given our rep of $\mathbb{R}$

$$\rho_k(t) = e^{kt} \qquad (\text{on } V = \mathbb{C})$$

what are ρ_k^* and $\bar{\rho_k}$?

$$\bar{\rho}_k(t)\bar{v} = \overline{\rho_k(t) v} \qquad \text{for } v \in \mathbb{C},\ \bar{v} \in \bar{\mathbb{C}}$$

just defn of "bar"

$v \in \mathbb{C}$, $\bar{v} \in \mathbb{C}$, $t \in \mathbb{R}$

$$\overline{\rho_K}(t)\,\bar{v} = \overline{\rho_K(t)\,v}$$

$$= \overline{e^{Kt}v} \qquad e^{Kt} \text{ is a } \#$$

$$= \overline{e^{Kt}}\cdot\bar{v}$$

$$= e^{\bar{K}t}\,\bar{v}$$

So, $\boxed{\overline{\rho_K}(t) \cong \rho_{\bar{K}}(t)}$ i.e. $\boxed{\overline{\rho_K} \cong \rho_{\bar{K}}}$

ρ_K^* is a rep of $\mathbb{R}$ on $\mathbb{C}^* \cong \mathbb{C}$ defined as:

$\mathbb{C}^* = \{f : \mathbb{C} \to \mathbb{C} \mid f \text{ lin.}\}$ think of this as mult by a #.

$$\left(\rho_K^*(t)(f)\right)(v) = f\left(\rho_K(-t)\,v\right) \qquad f \in \mathbb{C}^*,\ t \in \mathbb{R},\ v \in \mathbb{C}.$$

$$= f\left(e^{-Kt}v\right)$$

f is linear, $e^{Kt} = \#$.

$$= e^{-Kt}f(v)$$

So — $\boxed{\rho_K^*(t)f = e^{-Kt}f}$

So — $\boxed{\rho_K^* \cong \rho_{-K}}$

When is $\rho_K^* \cong \overline{\rho_K}$? when $-K = \overline{K}$.

i.e. K is purely imaginary.

$K = iE, \ E \in \mathbb{R}$

In this case, we have

$$\rho_K(t) = e^{iEt}$$ this is mult by a # on unit circle $(U(1))$

so $\rho_K(t) = e^{iEt} \in U(1)$.

We thus see that ρ_K is <u>unitary</u> in this case.

** Reps st dual & conjugate are equiv. are unitary **

Recall — In general, ρ unitary $\Rightarrow \rho^* \cong \overline{\rho}$.

Above we see the converse.

<u>Moral</u>: One-dim'l unitary reps of $\mathbb{R}$ are in 1-1 correspondence w/ $E \in \mathbb{R}$.
If we think of $\mathbb{R}$ as "time translation" then E is "energy".

i.e. In QM, if we have a particle of <u>energy</u> E whose state is some vector $\psi \in H$.
(H is some Hilbert space) and we evolve it in time for time t (ie wait for time t) its state becomes $e^{iEt}\psi$.

Energy is frequency!

(Time passing — we think of as the Real line.)