1/5/02

All 1-dim'le (smooth) reps of \((IR, +, 0)\) are equiv. to reps of this form

\[
\rho_K(t) = e^{kt}, \quad t \in IR
\]

We've seen \(\rho_K^* = \rho_{-K}\) and \(\overline{\rho_K} = \rho_{-K}\).

What about \(\rho_K \otimes \rho_L\)?

This is a rep. on \(C \otimes C \cong C\).

\[
(\rho_K \otimes \rho_L)(t) := \rho_K(t) \otimes \rho_L(t) : C \otimes C \to C \otimes C
\]

So this is just a linear map from \(C\) to \(C\) — so must, by a number.

In general, say we have \(A : C^n \to C^{n'}\) and \(B : C^m \to C^{m'}\). Then we get

\[
A \otimes B : C^n \otimes C^m \to C^{n'} \otimes C^{m'}
\]

\[
A e_i = \sum_j A_{ij} e_j
\]

\[
B f_k = \sum_l B_{kl} f'_l
\]

\[
\begin{cases}
    e_i \in C^n, & e_i' \in C^{n'} \\
    f_k \in C^m, & f'_l \in C^{m'}
\end{cases}
\]

Bases
(A \otimes B)(e_i \otimes f_k) = e_i \otimes f_k \in C^n \otimes C^m

Ae_i \otimes Bf_k = 

\sum_{j,l} A^i_j B^l_k e_j \otimes f'_l

\{the\ matrix\ description\ of\ A \otimes B\}

E.g. \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} u & v \\ w & x \end{pmatrix}

A \otimes B = \begin{pmatrix}
au & av & bu & bv \\
aw & ax & bw & bx \\
cu & cv & du & dv \\
cw & cx & dw & dx
\end{pmatrix}

And for us, \( p_k \), \( p_e \) are just 1-dim 'reps, so 1x1 matrices.

So - \( p_k(t) \) is the 1x1 matrix \( e^{kt} \)

\( p_e(t) \) is the 1x1 matrix \( e^{et} \)

so \( p_k(t) \otimes p_e(t) \) is the 1x1 matrix \( e^{kt} e^{et} = e^{(k+e)t} \)

So - \( p_k \otimes p_e \equiv p_{k+e} \)
What are all the intertwiners $T$ from $p_k$ to $p_e$?
Suppose $T: C \rightarrow C$ is an intertwiner from $p_k$ to $p_e$. Then:

$$T_{p_k}(t) = p_e(t)T$$  $\forall t \in \mathbb{R}$

$$\Rightarrow T e^{kt} = e^{kt} T$$  $\forall t \in \mathbb{R}$

But $T$ is just a number, i.e., $Tv = \alpha v$ for some $\alpha \in C$.
So we want all solns to

$$\alpha e^{kt} = e^{kt} \alpha$$  $\forall t \in \mathbb{R}$

If $k = 0$ then any $\alpha$ will satisfy this.
If $k \neq 0$ only $\alpha = 0$ will satisfy this.

(An example of Schur's lemma.)

What are all the intertwiners $T$ from

$$p_{k_1} \otimes \ldots \otimes p_{k_n} \rightarrow p_{l_1} \otimes \ldots \otimes p_{l_m}$$

So our quest is equiv. to knowing all intertwiners $S$ from

$$p_{k_1 + \ldots + k_n} \rightarrow p_{l_1 + \ldots + l_m}$$
(This is conservation of energy.)

By the above calculation we can only have \( T = 0 \) unless

\[ (*) \quad k_1 + \ldots + k_n = e_1 + \ldots + e_n \quad (\text{conservation law}) \]

and if this eqn. holds there's a 1-dim'l space of intertwiners: \( S \) can be any constant \( \alpha \) times \( I \).

We've seen \( p_k \) is unitary iff \( K = iE \), for some "energy" \( E \in \mathbb{R} \).

So \( (*) \) says sum of energies going in is same as going out.

Thus we get \underline{conservation of energy}:

\[ E_1 + \ldots + E_n = E_1' + \ldots + E_n' \]

Typical intertwiner in rep. theory of \( R \):

We have energy conservation at each vertex.

"Energy network"
Another way to get new representations from old ones — direct sum:

Given any rep $\rho : G \rightarrow \text{GL}(V)$

$\rho' : G \rightarrow \text{GL}(V')$

we get a rep

$\rho \oplus \rho' : G \rightarrow \text{GL}(V \oplus V')$

by $(\rho \oplus \rho')(g)(v, v') = (\rho(g)v, \rho'(g)v')$

where $(v, v') \in V \oplus V'$

Should check this is a rep.

\[\begin{array}{cccc}
\text{Set} & \text{Logic} & \text{Vect}_\mathbb{C} \\
\text{Arithmetic} & \text{Set Theory} & \text{Linear Alg} \\
\text{over } \mathbb{C} \\
\end{array}\]

\[\begin{array}{cccc}
+ & \uparrow & \odot & \text{false} \\
\text{"disjoint union"} & \text{or} & \text{"Cartesian product"} & \text{"the zero } \mathbb{C} \text{-space"}
\end{array}\]

\[\begin{array}{cccc}
\emptyset & \text{false} & \text{true} & \text{true} \\
\text{"the one-element set"} & \text{false} & \text{true} & \text{true}
\end{array}\]
So any set $X$ has $X \times \{1\} \cong X$

\[ (x, y) = \{\{(x, 1), \{(x, y)\}\} \]

So in QM we use a Hilbert space to describe the states of a system; the states will be described by unit vectors in the Hilbert space.

Ex - suppose system is particle on the unit interval:

\[
\begin{align*}
\text{Hilbert space: } & L^2([0,1]) \\
\text{a particle on } & [0,1] \\
L^2([0,1]) \oplus L^2([0,1]) & \text{ Hilbert space for a particle on } 1\text{st copy of } [0,1] \text{ or } 2\text{nd copy of } [0,1].
\end{align*}
\]

The condition that $y \in L^2([0,1]) \cup [2,3]$ has $\|y\|=1$ says

\[
\int_0^1 |y|^2 \, dx + \int_2^3 |y|^2 \, dx = 1
\]
Note: $L^2(X) = \text{Hilb}$, space for a particle moving around in $X$

\[
\int_0^1 |4x|^2 \, dx + \int_2^3 |4x|^2 \, dx = 1
\]

Probability that the particle is in $[0,1]$; probability that the particle is in $[2,3]$; which says the particle is in one box or the other.

So we see how:

+ Hilb spaces
\(\perp\) measure spaces
\(\lor\) probability
\(\land\) logic

are all related.

Similarly for tensor product:

\[
L^2([0,1]) \otimes L^2([0,1]) \overset{\text{Hilbert}}{\longrightarrow} L^2([0,1] \times [0,1])
\]

Hilbert space for a particle in the square $[0,1] \times [0,1]$ or Hilbert space for 2 particles; one in $[0,1]$ and another particle in $[0,1]$. w/ $f \circ g$, $f \otimes g$.