11/5/02

All 1-dim'l (smooth) reps of $(\mathbb{R}, +, 0)$ are equiv. to reps of this form

$$\rho_k(t) = \underset{\substack{\cap \\ GL(\mathbb{C})}}{e^{kt}} \qquad t \in \mathbb{R}$$

where $k \in \mathbb{C}$.

We've seen $\boxed{\rho_k^* \cong \rho_{-k}}$ and $\boxed{\bar{\rho}_k = \rho_{\bar{k}}.}$

What about $\rho_k \otimes \rho_\ell$?

This is a rep. on $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$.

$$(\rho_k \otimes \rho_\ell)(t) := \rho_k(t) \otimes \rho_\ell(t) : \underset{\substack{\cong \\ \mathbb{C}}}{\mathbb{C} \otimes \mathbb{C}} \longrightarrow \underset{\substack{\cong \\ \mathbb{C}}}{\mathbb{C} \otimes \mathbb{C}}$$

So this is just a linear map from $\mathbb{C}$ to $\mathbb{C}$ — so mult. by a number.

In general, say we have $A : \mathbb{C}^n \longrightarrow \mathbb{C}^{n'}$ and $B : \mathbb{C}^m \longrightarrow \mathbb{C}^{m'}$. Then we get

$$A \otimes B : \mathbb{C}^n \otimes \mathbb{C}^m \longrightarrow \mathbb{C}^{n'} \otimes \mathbb{C}^{m'}$$

$$A e_i = \sum_j A^j_i \, e_j' \qquad \left\{ \begin{array}{l} e_i \in \mathbb{C}^n, \; e_i' \in \mathbb{C}^{n'} \\ \\ f_k \in \mathbb{C}^m, \; f_\ell' \in \mathbb{C}^{m'} \end{array} \right.$$

$$B f_k = \sum_\ell B^\ell_k f_\ell'$$

Bases

tensoring lin. transf.

$$(A\otimes B)(e_i \otimes f_k) = \qquad e_i \otimes f_k \in \mathbb{C}^n \otimes \mathbb{C}^m$$

$$Ae_i \otimes Bf_k =$$

$$\sum_{j,\ell} \underbrace{A_i^{\,j} B_k^{\,\ell}}_{\text{the matrix description of } A\otimes B} e_j' \otimes f_\ell'$$

E.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad B = \begin{pmatrix} u & v \\ w & x \end{pmatrix}$

$$A\otimes B = \left(\begin{array}{cc|cc} au & av & bu & bv \\ aw & ax & bw & bx \\ \hline cu & cv & du & dv \\ cw & cx & dw & dx \end{array}\right)$$

And for us, ρ_k, ρ_ℓ are just 1-dim'l reps, so 1×1 matrices.

So – $\rho_k(t)$ is the 1×1 matrix e^{kt}

$\rho_\ell(t)$ is the 1×1 matrix $e^{\ell t}$

so $\rho_k(t) \otimes \rho_\ell(t)$ is the 1×1 matrix $e^{kt}e^{\ell t} = e^{(k+\ell)t}$

So – $\boxed{\rho_k \otimes \rho_\ell \cong \rho_{k+\ell}}$

What are all the intertwiners from ρ_k to ρ_ℓ?
Suppose $T : \mathbb{C} \longrightarrow \mathbb{C}$ is an intertwiner from ρ_k to ρ_ℓ. Then:

$$T\rho_k(t) = \rho_\ell(t)\,T \qquad \forall\, t \in \mathbb{R}$$

$$\Rightarrow \qquad Te^{kt} = e^{\ell t}\,T \qquad \forall\, t \in \mathbb{R}$$

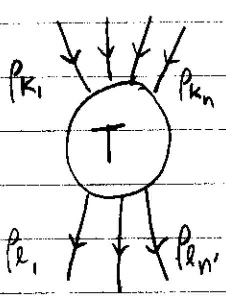
But T is just a number. i.e. $Tv = \alpha v$ for some $\alpha \in \mathbb{C}$.
So we want all solns to

$$\alpha e^{kt} = e^{\ell t}\alpha \qquad \forall\, t \in \mathbb{R}$$

If $k = \ell$ then any α will satisfy this.
If $k \neq \ell$ only $\alpha = 0$ will satisfy this.

(An example of Schur's lemma)

What are all the intertwiners T from



$$\begin{tikzcd}
\rho_{k_1} \otimes \cdots \otimes \rho_{k_n} \arrow[r, "T"] \arrow[d, "\wr"'] & \rho_{\ell_1} \otimes \cdots \otimes \rho_{\ell_{n'}} \arrow[d, "\wr"] \\
\rho_{k_1+\cdots+k_n} \arrow[r, "S"] & \rho_{\ell_1+\cdots+\ell_{n'}}
\end{tikzcd}$$

So our quest is equiv. to knowing all intertwiners S from

$$\rho_{k_1+\cdots+k_n} \quad \text{to} \quad \rho_{\ell_1+\cdots+\ell_{n'}}.$$

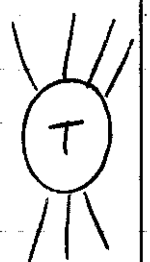
(This is conservation of energy!)

By the above calculation we can only have $T = 0$ unless

(*) $\quad k_1 + \dots + k_n = \ell_1 + \dots + \ell_{n'}$ $\quad$ (conservation law)

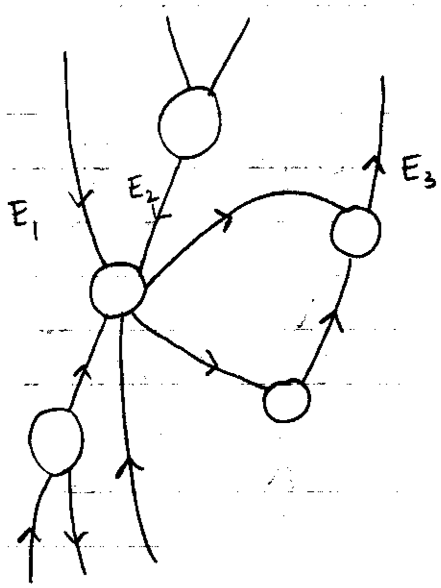
and if this eqn. holds there's a 1-dim'l space of intertwiners: S can be any constant α times I.

We've seen ρ_k is unitary iff $k = iE$ for some "energy" $E \in \mathbb{R}$.



So (*) says sum of energies going in is same as going out.
Thus we get <u>conservation of energy</u>.

$$E_1 + \dots + E_n = E_1' + \dots + E_{n'}'.$$



Typical intertwiner in rep. theory of $\mathbb{R}$.

We have energy conservation at each vertex.

"Energy network"

Another way to get new representations from old ones — direct sum:

Given any grp G and reps:

$$\rho : G \longrightarrow GL(V)$$

$$\rho' : G \longrightarrow GL(V')$$

we get a rep

$$\rho \oplus \rho' : G \longrightarrow GL(V \oplus V')$$

by $\quad (\rho \oplus \rho')(g)(v, v') = (\rho(g)v, \rho'(g)v')$

where $(v, v') \in V \oplus V'$

Should check this is a rep.

$\mathbb{C}$ Arithmetic	Set Set Theory	Logic	$\text{Vect}_{\mathbb{C}}$ Linear Alg (over $\mathbb{C}$)
$+$	$\coprod$ "disjoint union"	$\vee$ or	$\oplus$
$\times$	$\times$ "Cartesian product"	$\wedge$ and	$\otimes$
0	$\emptyset$	F false	$\{0\}$ 'the zero v. space'
1	any one elt. set 1 "the one-elt. set"	T true	$\mathbb{C}$

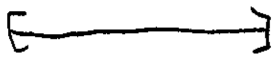
so any set X has $X \times 1 \cong X$

↑ the one elt set

$(x,y) = \{\{x\}, \{\{x\}, y\}\}$

So— in QM we use a Hilbert space to describe the states of a system; the states will be described by unit vectors in the Hilbert space.

Ex— suppose system is particle on the unit interval:



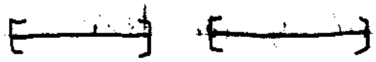
Hilbert space: $L^2([0,1])$

a particle on $[0,1]$

$|\psi|^2(x)$ is prob. density of finding particle at x.

$L^2[0,1] \oplus L^2[0,1]$

$\cong$



$L^2([0,1] \amalg [0,1])$

$\cong$

Hilbert space for a particle on 1^{st} copy of $[0,1]$ or 2^{nd} copy of $[0,1]$.

$L^2([0,1] \amalg [2,3])$

The condition that $\psi \in L^2([0,1] \cup [2,3])$ has $\|\psi\| = 1$ says

$$\int_0^1 |\psi|^2 dx + \int_2^3 |\psi|^2 dx = 1$$

<u>Note</u>: $L^2(X)$ = Hilb. space for a particle moving around in X

$$\underbrace{\int_0^1 |\psi|^2\,dx}_{} + \underbrace{\int_2^3 |\psi|^2\,dx}_{} = 1$$

probability that particle is in $[0,1]$

probability that particle is in $[2,3]$

which says the particle is in one box <u>or</u> the other. So we see how

$\oplus$ Hilb spaces
$\amalg$ measure spaces
$+$ probability
$\vee$ logic

are all related.

Similarly for tensor product:

$$L^2([0,1]) \otimes L^2([0,1]) \qquad f\otimes g$$
$$\cong \qquad\qquad \downarrow$$
$$L^2([0,1]\times[0,1]) \qquad fg$$

w/ $fg(x,y) = f(x)\,g(y)$

Hilbert space for a particle in the square $[0,1]\times[0,1]$ <u>or</u> Hilbert space for 2 particles: one in $[0,1]$ <u>and</u> another particle in $[0,1]$.