have a rep on 0-dim'l v.space — acts as identity

11/12/02

<u>Defn</u>: A rep $(\rho, V)$ of a group $G$ is <u>indecomposable</u> if it's not equivalent to a direct sum of 2 nontrivial reps unless exactly one of those reps is 0-dim'l. (rule out case that we write something as a sum of 0 w/ itself)

<u>Note</u>: $(\rho, V) \cong (\rho, V) \oplus (\rho_{\{0\}}, \{0\})$

<u>Defn</u>: A rep $(\rho, V)$ of $G$ has a rep $(\sigma, U)$ as a <u>subrepresentation</u> if $U \subseteq V$ and

$\rho(g): U \longrightarrow U \quad \forall g$ so that we can define

$$\sigma(g) = \rho(g)\big|_U : U \longrightarrow U.$$

<u>Defn</u>: A rep $(\rho, V)$ is <u>irreducible</u> iff it has no subrepresentations except $(\rho|_{\{0\}}, \{0\})$

and $(\rho, V)$ itself, (which must be different)

<u>Note</u>: We don't want the zero rep to be indecomposable or irreducible.
(Just as 1 isn't prime.) (If it were — wouldn't have unique decomposition).

So — we exclude the zero-dim'l rep from being indecomposable or irreducible.

Note: irreducible $\Rightarrow$ indecomposable

Thm: If $(\rho, \mathcal{H})$ is a unitary rep, $\rho$ is indecomposable iff $\rho$ is irreducible.

proof: In general, $\rho$ irreducible $\Rightarrow$ $\rho$ indecomposable.

For converse — we use the fact $\rho$ is unitary:

Suppose $\rho$ has a subrep $(\rho|_U, U)$ (other than $\{0\}$ or $V$) and show that

$U^\perp$ (all vectors $\perp$ to vectors in $U$) is preserved by $\rho(g)$ $\forall g \in G$. We'll show $V = U \oplus U^\perp$.

$v \in U^\perp \Rightarrow \langle v, w \rangle = 0 \quad \forall w \in U.$

Now, consider $\langle \rho(g) v, w \rangle = \langle v, \rho(g)^{-1} w \rangle = \langle v, \underbrace{\rho(g^{-1}) w}_{} \rangle$

since $\rho$ is unitary, so $\rho(g)^* = \rho(g)^{-1}$

this is in $U$ since $U$ is a subrep.

$= 0.$

Thus, $\rho(g) v \in U^\perp$.

Therefore – $(\rho|_{U^\perp}, U^\perp)$ is also a subrep.

of $(\rho, V)$ and so $\boxed{(\rho, V) \cong (\rho|_{U^\perp}, U^\perp) \oplus (\rho|_U, U)}$

$\blacksquare$

Want an example of indecomposable but not irreducible.
(can't be unitary!)

<u>Example</u>: What's a rep of $(\mathbb{R}, +)$ which is indecomposable but not irreducible?

$$\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ is a rep of } \mathbb{R} \text{ on } \mathbb{C}^2.$$

(upper $\Delta$-ular matrix won't be unitary)

<u>Note</u>: This matrix isn't unitary for <u>any</u> inner product.

The subspace $\mathcal{U} = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \middle| \, a \in \mathbb{C} \right\} \subseteq \mathbb{C}^2$

is preserved by $\rho(t)$ $\forall t$, so it's a subrep, and thus our rep is reducible.
(neither $\{0\}$ nor $\mathbb{C}^2$.)
(trivial rep – $\rho(t)$ is identity)

But it's not decomposable. In particular $\nexists$ $\mathcal{U}' \subseteq \mathbb{C}^2$ which is also preserved by $\rho(t)$ $\forall t$ and st $\mathbb{C}^2 \cong \mathcal{U} \oplus \mathcal{U}'$.

Just need to check that the only subreps are $\{0\}$, $\mathcal{U}$, and $\mathbb{C}^2$. $\boxtimes$

∴ $\rho_k$ are all 1-dim'l reps on $\mathbb{R}$.

recall: $\rho_k(t) = e^{kt}$

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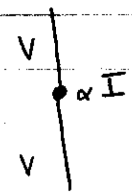



unitary means $|\alpha| = 1$.

<u>Schur's Lemma</u>: If a rep $(\rho, V)$ is irreducible and $T : V \longrightarrow V$ is an intertwiner then $T = \alpha 1_V$ for some $\alpha \in \mathbb{C}$.

<u>proof</u>: Suppose $T$ is an intertwiner. Then any eigenspace

$$U = \{ v \in V \mid Tv = \lambda v \} \text{ is a subrep.}$$

Let $v \in U$. Then $Tv = \lambda v$.

T an eigenvector

$$\Rightarrow T\rho(g)v = \rho(g)Tv = \lambda \rho(g)v$$

because T an intertwiner

Thus, $\rho(g)v \in U$.

But since $U$ must be $\{0\}$ or $V$, $T$ has at most one eigenvalue, so $T = \alpha I$. ∎

<u>Corollary</u>: If $G$ is abelian, then all its irreps (irreducible reps) are 1-dim'l.

pf: Suppose $(\rho, V)$ is an irrep of $G$.
Since $G$ is abelian:

$$\rho(g)\rho(g') = \rho(gg') = \rho(g'g) = \rho(g')\rho(g) \qquad \forall\, g, g' \in G$$

(a homomorphism maps an abelian grp into an abel. subgrp)

But — we call an intertwiner something that commutes w/ $\rho(g)$ $\forall g$.

So — $\rho(g)$ is an intertwiner $\forall\, g \in G$.

By Schur's lemma, $\rho(g) = \alpha_g I \quad \forall\, g \in G$.
(multiple depends on what $g$ is)

Then — $T: V \longrightarrow V$ is an intertwiner if $T$ commutes w/ $\rho(g)$ — that is $T$ is an intertwiner <u>for all</u> $T: V \longrightarrow V$ because $\rho(g)$ is a mult of identity.

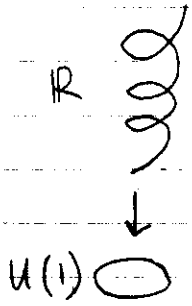
By Schur's lemma, all $T: V \longrightarrow V$ must be multiples of $I$. So — $V$ must be 0-dim'l or 1-dim'l. But, 0-dim'l ones aren't irreducible, so since $V$ is irred. it must be 1-dim'l. ☑

<u>Corollary</u>: Every (smooth) irrep of $\mathbb{R}$ is of the form

$$\rho_k(t) = e^{kt}.$$

<u>pf</u>: We know all the 1-dim'l (smooth) reps are of this form; since they're 1-dim'l they're irreducible and there are no higher-dim'l irreps by previous corollary. ☑

How about $U(1)$? We have an onto homo




$$\begin{tikzcd} \mathbb{R} \arrow[d, "\rho"'] & t \arrow[d, mapsto] \\ U(1) & e^{it} \end{tikzcd}$$

so any rep of $U(1)$ gives a rep of $\mathbb{R}$:

$$\begin{tikzcd} \mathbb{R} \arrow[d, "\rho"'] \arrow[dr, dashed] & \\ U(1) \arrow[r, "\rho"'] & GL(V) \end{tikzcd}$$

and any irrep of $U(1)$ must give an irrep of $\mathbb{R}$.

So – the only options for $\rho$ are:

$$\begin{tikzcd} \mathbb{R} \arrow[d, "\rho"'] \arrow[dr, "\rho_k"] & \\ U(1) \arrow[r, "\rho"'] & GL(\mathbb{C}) \end{tikzcd}$$

(commutes)

i.e. $\rho(e^{it}) = \rho_k(t) = e^{kt}$ $\quad t \in \mathbb{R}$, $k \in \mathbb{C}$

Since $\rho(e^{2\pi i}) = 1$ we need

$$e^{k \cdot 2\pi} = 1 \quad \text{so} \quad ik \in \mathbb{Z}$$

reps of $\mathbb{Z}$ form grp $U(1)$ (by Poincaré duality)

<u>Thm</u>: All irreps of $U(1)$ are of the form

$$\rho(e^{it}) = e^{int} \quad \text{for some } n \in \mathbb{Z}.$$

<u>Note</u>: $U(1)$, $\mathbb{R}$ are 'boring' since both are 1-dim'l & abelian.

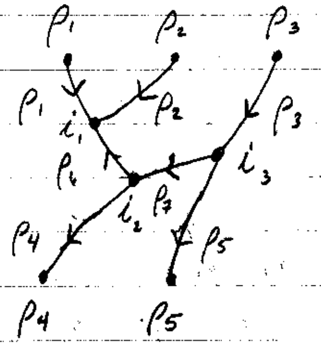
| Compact Lie grps | Noncompact Lie grps |
|---|---|
| countably many smooth irreps, all finite-dim'l | typically uncountably many (smooth) irreps, |
| | typically some infinite-dim'l, |
| all are unitary w/r/t some inner product | typically many can't be made unitary. |
| e.g.: $\underset{U(1)}{\rho}(e^{it}) = e^{int}$ $n \in \mathbb{Z}$ | e.g.: $\underset{\mathbb{R}}{\rho}(t) = e^{kt}$, $k \in \mathbb{C}$ |

11/14/02

Let $G$ be a (Lie) group. There's a category Net$(G)$ where the objects are lists of irreducible (smooth) unitary reps of $G$:

$$\rho_1 \quad \rho_2 \quad \rho_3 \quad \rho_4$$

The morphisms are (certain equiv. classes) of diagrams like:




vertices of the graph are the ones in the middle, not top & bottom.

label vertices w/ $i$'s: intertwiners

$$i_1 : \rho_1 \otimes \rho_2 \otimes \rho_6 \longrightarrow \mathbb{C}$$

no arrows going out!

$$i_2 : \rho_7 \longrightarrow \rho_6 \otimes \rho_4$$

$$i_3 : \rho_3 \longrightarrow \rho_7 \otimes \rho_5$$

<u>Note</u>: edges are labelled by unitary irreps; vertices labelled by intertwiners.
Edges are directed; changing direction and replacing rep by its dual doesn't change diagram.

Incoming & outgoing edges, if directed downwards, must be labelled by reps that match those on corresponding dots at top & bottom.

We get a symmetric braided monoidal category w/ duals.

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In quantum field theory we call a morphism in Net(G) a "Fenyman diagram", edges represent "particles" and vertices represent "interactions."

* We take a diagram (like on prev pg) and compose & tensor the intertwiners $i$'s and get a big intertwiner from $\rho_1 \otimes \rho_2 \otimes \rho_3 \longrightarrow \rho_4 \otimes \rho_5$.

What does it mean to "evaluate" a Fenyman diagram?

We take one and compose/tensor all the intertwiners to get a new intertwiner:

$$i: (\rho, v) \longrightarrow (\rho', v')$$

$$\rho \longrightarrow \rho'$$

(linear map bet. the v. spaces)

where $\rho$ is the tensor product of all the irreps labelling the incoming edges; $\rho'$ is the tensor product of reps labelling outgoing edges.

So, we turn objects in Net(G) into reps
morphisms in Net(G) into intertwiners

That is – we get a <u>functor</u>:

$$\text{eval}: \text{Net}(G) \longrightarrow \text{Rep}(G)$$

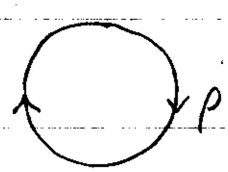
where Rep(G) is the category whose objects are unitary (smooth) reps of G and whose morphisms are intertwiners.

<u>Note</u>: Can't always compose / tensor intertwiners and get a new one — we need finite dim'l.

* Actually this "eval" is well-defined if we restrict attention to <u>finite-dim'l</u> unitary reps when defining Net(G) and Rep(G).

Otherwise the evaluation of a diagram may "diverge."

$\rho^*$ is $\rho$ w/ arrow pointing up



$$\begin{array}{c} \mathbb{C} \\ \downarrow \\ \rho^* \otimes \rho \\ \downarrow \\ \mathbb{C} \end{array}$$

mult by some #.

The linear map from $\mathbb{C}$ to $\mathbb{C}$ is ill-defined if dim of our rep is infinite.

We're talking about unitary reps! (Hilbert spaces)

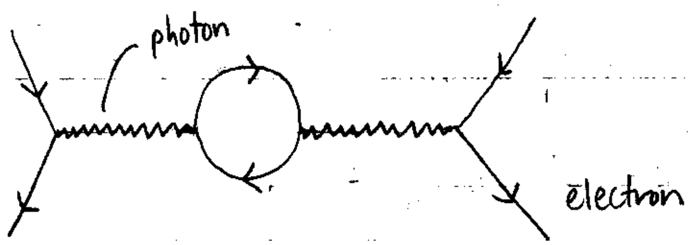
Say our rep is $(\rho, H)$ and Hilbert space $H$ has orthonormal basis $e_i$ and $H^*$ has dual basis $e^i$. Then:

$$\begin{array}{ccc} & \mathbb{C} & 1 \\ \cap & \downarrow & \downarrow \\ & H^* \otimes H & \sum_i e^i \otimes e_i \\ \cup & \downarrow & \downarrow \\ & \mathbb{C} & \sum_i \underbrace{e^i(e_i)}_{=1} = \sum_i 1 = \dim(H) \end{array}$$

since $i$ is running over a basis.

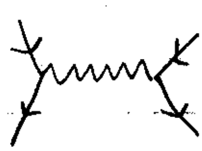
When $H$ is infinite dim'l, this blows up.

In general — whenever you allow infinite dim'l reps to label edges &, try to evaluate a Fenyman diagram which are not simply-connected, you run the risk of having the evaluation diverge:

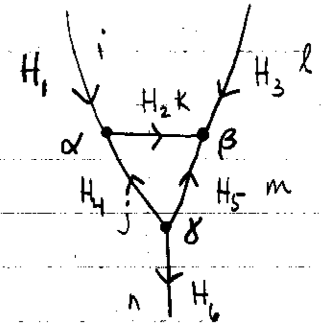



2 electrons exchange a photon

But simply-connected diagrams are okay:



Example:




$\alpha : H_1 \otimes H_4 \longrightarrow H_2$

$\beta : H_3 \otimes H_2 \otimes H_5 \longrightarrow \mathbb{C}$

$\gamma : \mathbb{C} \longrightarrow H_4 \otimes H_5 \otimes H_6$

want to put these together to get something from

$$\iota : H_1 \otimes H_3 \longrightarrow H_6.$$

$\alpha^{ij}_{k}$ is a matrix elt. for $i$ running over basis of $H_1$

$j$ " " $H_4$

$k$ " " $H_2$

Similarly we get: $\beta^{klm}$ and $\gamma_{jmn}$

Then we multiply matrices: actually tensors

$$\sum_{k,j,m} \alpha^{ij}_{k} \beta^{klm} \gamma_{jmn} = \iota^{il}_{n}$$

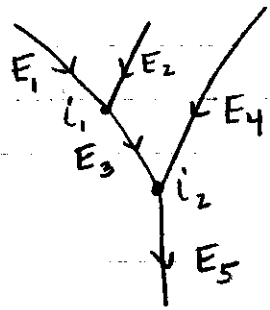
* the edges we're summing over are the "inside" ones.

The answer is basis-independent!

Some examples:

1) $G = \mathbb{R}$. Then unitary irreps. are 1-dim'l:
$$\rho_E(t) = e^{iEt} \quad t \in \mathbb{R},\ E \in \mathbb{R}$$

We call $E$ the "energy" and our diagrams are "energy networks":




For there to be an intertwiner at each vertex, we need "conservation of energy"

$$\underbrace{E_1 + E_2}_{\text{in}} = \underbrace{E_3}_{\text{out}} \text{ for } i_1.$$

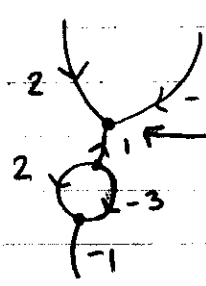
* We think of $\mathbb{R}$ as time translation, and the conserved quantity is energy.

2) $G = U(1)$ "gauge grp for electromagnetism"

"phase changes" in electromagnetism

Then the unitary irreps $\rho_q(e^{it}) = e^{iqt}$ $\quad e^{it} \in U(1)$, $q \in \mathbb{Z}$

We call $q$ the "charge" and our diagrams are "charge networks":




now we need "conservation of charge."

3) $G = \mathbb{R}^4$

spacetime translations

Here irreps are labelled by 4 numbers:

$$(\underbrace{E}_{\substack{= p_t \\ \text{energy}}}, \underbrace{p_x, p_y, p_z}_{\text{momentum in } x, y, z \text{ direction}}) \in \mathbb{R}^4$$

need
$p = p' + p''$
for there to
be an intertwiner
- conservation of
"energy-momentum"