11/19/02

<u>Example</u>: Let $G = \mathbb{R}^4$ (spacetime translations)

Notice

$$\mathbb{R}^4 \subseteq SO_0(3,1) \ltimes \mathbb{R}^4 = P \quad \text{(Poincaré grp)}$$

If we have two groups, say G & H and we have a rep (ρ, V) of G and a rep (ρ', V') of H we can get a rep $(\rho \otimes \rho', V \otimes V')$ of $G \times H$:

$$(\rho \otimes \rho')(g,h) = \rho(g) \otimes \rho'(h) \qquad g \in G,\ h \in H$$

(we've already talked about how to start w/ 2 reps ρ, ρ' of G so that $\rho \otimes \rho'$ is a rep of G also.)

This is related to, but different from, the trick for taking 2 reps (ρ, V), (ρ', V') of G and getting the rep $(\rho \otimes \rho', V \otimes V')$ of G:

$$(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g).$$

Knowing reps of $\mathbb{R}$ means we know reps of products of $\mathbb{R}$ w/ itself.

<u>Thm</u>: If (ρ, V) is a (unitary) irrep of G and (ρ', V') is a (unitary) irrep of H then $(\rho \otimes \rho', V \otimes V')$ is a (unitary) irrep of $G \times H$.

Moreover — <u>every</u> (unitary) irrep of $G \times H$ is equivalent to one of this sort.

<u>Note</u>: we saw all irreps of $\mathbb{R}$ were e^{kt}, $k \in \mathbb{C}$

now write as e^{ikt}, $k \in \mathbb{R}$

<u>Corollary</u>: Every irrep of $\mathbb{R}^4$ is of the form

$$\rho_{k_0} \otimes \rho_{k_1} \otimes \rho_{k_2} \otimes \rho_{k_3} \quad \text{where } k = (k_0, k_1, k_2, k_3) \in \mathbb{R}^4$$

and ρ_k is the irrep of $\mathbb{R}$ given by

$$\rho_k(t) = e^{kt} \qquad t \in \mathbb{R},\ k \in \mathbb{C}$$

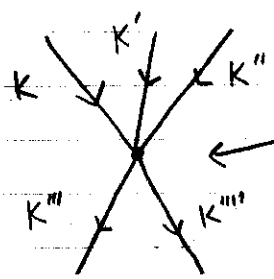
The unitary irreps of $\mathbb{R}^4$ are of the form

$$\rho_{k_0} \otimes \rho_{k_1} \otimes \rho_{k_2} \otimes \rho_{k_3} \quad \text{where}$$

$$k = i(k_0, \ldots, k_3), \quad (k_0, \ldots, k_3) \in \mathbb{R}^4$$

---

we call $(\underset{\overset{\shortparallel}{E}}{k_0}, \underbrace{k_1, k_2, k_3}_{\vec{k}}) \in \mathbb{R}^4$ the "<u>energy-momentum vector</u>" of this rep.



there will be a 1-dim'l space of intertwiners iff

$$k + k' + k'' = k''' + k''''$$

conservation of energy-momentum

$SL(2,\mathbb{C})$ acts as itself on $\mathbb{C}^2$, also the conjugate of this action — give us $(\frac{1}{2},0)$, $(0,\frac{1}{2})$

Next examples:

not compact!

$$\begin{tikzcd}
SU(2) \arrow[r, hook] \arrow[d, "2-1"'] & SL(2,\mathbb{C}) \arrow[r, hook] \arrow[d, "2-1"] & \tilde{P} \arrow[d, "2-1"] \\
SO(3) \arrow[r, hook] & SO_0(3,1) \arrow[r, hook] & SO_0(3,1) \ltimes \mathbb{R}^4 = P
\end{tikzcd}$$

Irreps of $SU(2)$ are classified by "spins"

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

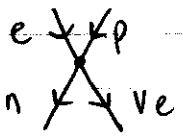
these are related to angular momentum. Finite-dim'l irreps of $SL(2,\mathbb{C})$ aren't unitary (except for 1-dim'l trivial rep - always unitary) and are classified by pairs of spins (j,k) — left-handed / right-handed spin.

not equivalent

$(\frac{1}{2},0)$ is the tautologous rep of $SL(2,\mathbb{C})$ on $V = \mathbb{C}^2$.
(defining)

$(0,\frac{1}{2})$ is the conjugate rep of $SL(2,\mathbb{C})$ on $\overline{V}$.

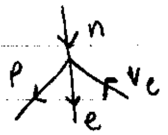
neutrinos & antineutrinos



fusion: $e + p \longrightarrow n + \nu_e$

electron + proton $\longrightarrow$ neutron + ν_e

(-) (+) (neutral) └ neutrino



$n \longrightarrow p + e + \bar{\nu}_e$

└ antineutrino

Unitary irreps of $\tilde{P}$ are complicated — most of them are classified by:

- $m \in \mathbb{R}$, $m > 0$: mass
- spin $j = 0, \frac{1}{2}, 1, \ldots$

But there are also unitary irreps w/ $m = 0$ and spin either $(j, 0)$ or $(0, j)$ - massless particles have a handedness to their spin.

There are also weirder unitary irreps

e.g. "tachyons" w/ mass being imaginary

In quantum electrodynamics we have 2 kinds of particles — i.e. reps of $\tilde{P}$

- photons $m = 0$, spin $(1,0) \oplus (0,1)$

  (approximately

- electron/positrons $m = m_e$ spin $= \frac{1}{2}$

  $= .511 \text{ MeV}/c^2$

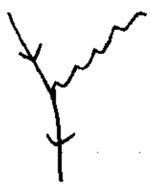
We draw these reps as:

 electron  photon

<u>Note</u>:

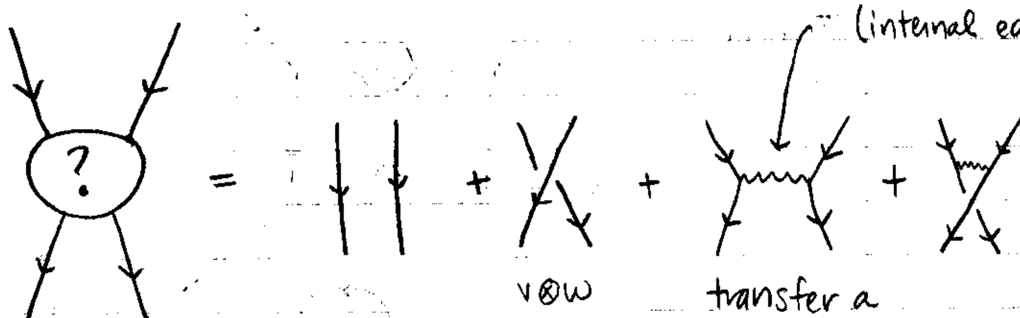
antiphoton = photon
(self-dual)

and the basic intertwiner is:



i.e. goes from electron/positron rep ⊗ photon rep
to electron/positron rep.

virtual photon
(internal edge)



$v \otimes w \mapsto -w \otimes v$

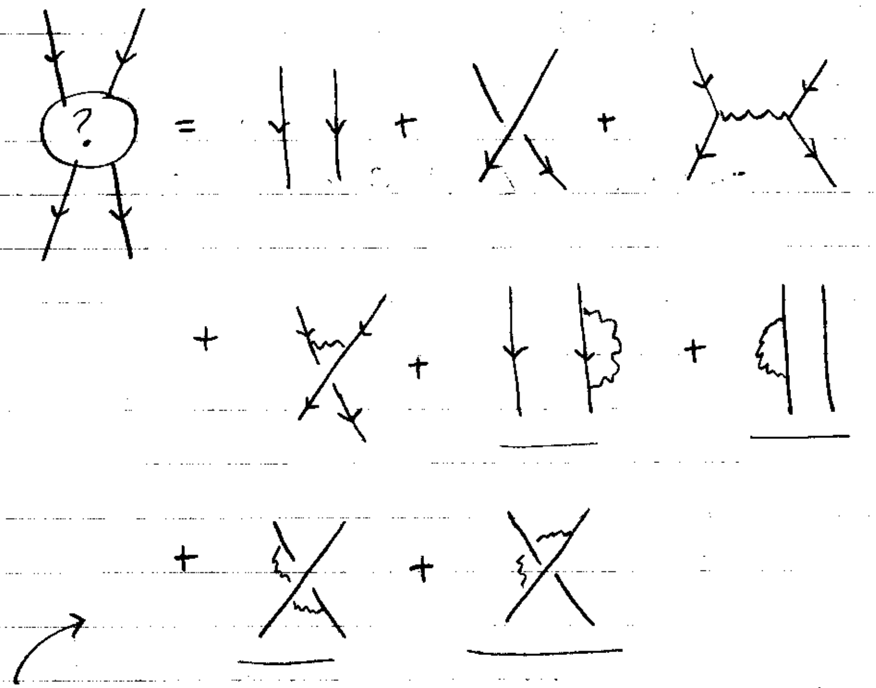
transfer a photon

don't know who emits the photon & who absorbs it!

Note - Braiding commutes w/ everything, so

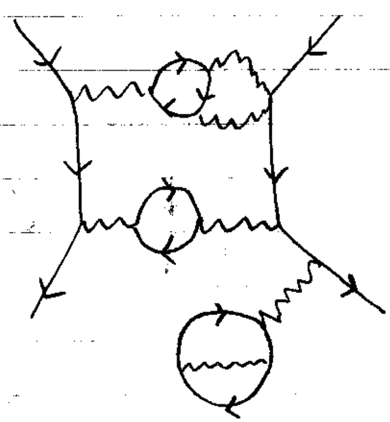
switch then photon

photon then switch



But last four aren't simply connected!

+ ....



The simply-connected ones above (1st four) give well-defined intertwiners; the others "diverge" so we need to invent clever "renormalization" tricks to extract intertwiners from them.

Unfortunately — the infinite sum also (probably) diverges.

So - one basic problem is: see if any interacting QFT in 4 dimensions makes rigorous sense!

11/21/02

Irreducible reps of $SU(2)$ and $SL(2,\mathbb{C})$.

$SU(2)$ and $SL(2,\mathbb{C})$ both have a 2-d "defining" rep on $\mathbb{C}^2$. It's irreducible since there aren't any 1-d subspaces of $\mathbb{C}^2$ invariant under all $SU(2)$. Let's write

$$V = \mathbb{C}^2$$

w/ this rep of $SU(2)$, $SL(2,\mathbb{C})$ on it.

What about V^*, $\overline{V}$? V^* is equivalent to V as a rep of $SL(2,\mathbb{C})$ and thus $SU(2) \subseteq SL(2,\mathbb{C})$. Why?

Want an intertwiner:

$$i: V \xrightarrow{\sim} V^*$$

(2 ways to take vectors & get a linear functional.)

1) inner product space
2) use symplectic structure

We can get this by defining

$$\begin{aligned} \omega: V \otimes V &\longrightarrow \mathbb{C} \\ v \otimes w &\longmapsto \omega(v \otimes w) := \omega(v,w) \end{aligned}$$

which is

1) <u>nondegenerate</u>: $\omega(v,w) = 0 \ \forall\, w \in V \Rightarrow v = 0$.

2) invariant under $SL(2,\mathbb{C})$:

$$\omega(gv, gw) = \omega(v,w) \quad g \in SL(2,\mathbb{C}).$$

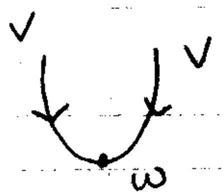
ω gives a map

$$i : V \longrightarrow V^*$$
$$v \longmapsto \omega(v, \cdot)$$

so since ω is nondegenerate, $\Rightarrow$ i is 1-1
moreover – since $\dim V = \dim V^* = 2$, i is onto.
$(V = \mathbb{C}^2)$.

The fact that i is an intertwiner is equivalent to the invariance of ω.

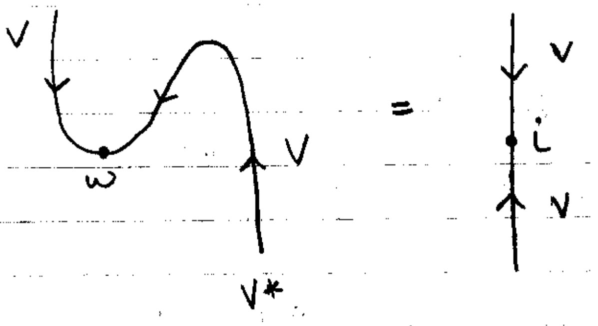
Note – invariance of ω says ω is an intertwiner from $V \otimes V$ to $\mathbb{C}$ (as a <u>trivial</u> rep of $SL(2,\mathbb{C})$).



$$V \otimes V \xrightarrow{\ \omega\ } \mathbb{C}$$

ω has 2 inputs,
i has 1 input, 1 output

So– take ω and turn an input into an output.



algebraically

$$\begin{array}{c} V \\ \wr\wr \\ V \otimes \mathbb{C} \\ \big\downarrow 1_V \otimes i_V \quad (\text{unit}) \\ V \otimes V \otimes V^* \\ \big\downarrow \omega \otimes 1_{V^*} \\ \mathbb{C} \otimes V^* \\ \wr\wr \\ V^* \end{array}$$

Since ω is an intertwiner and we get i from ω using diagram tricks, it's an intertwiner.

<u>Moral</u>: If (ρ, V) is a finite-dim'l rep of G and we have a nondegenerate invariant $\omega : V \otimes V \to \mathbb{C}$ then $(\rho^*, V^*) \cong (\rho, V)$.

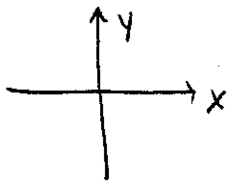
(determinant of transf.) — tells how volume is changed

det = 1 preserves volume

But — what's ω in our case?

$\omega : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}$

Area:



Area — $dx\,dy = dx \wedge dy$
so a transformation that preserves dx, dy preserves area

2-form — eats 2 vectors, gives a number
and that's what ω is!

Note — $g \in SL(2, \mathbb{C})$ means $g : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ has $\det(g) = 1$ which means g preserves volumes, or in 2 dimensions, preserves <u>areas</u>

i.e. preserves the area 2-form $dx \wedge dy$ where x, y are coordinates on $\mathbb{C}^2$.
- dx, dy are dual basis of $(\mathbb{C}^2)^*$ so $dx \wedge dy$ is a 2-form i.e. skew-symmetric bilinear map from $\mathbb{C}^2 \times \mathbb{C}^2$ to $\mathbb{C}$, so we let

$$\omega = dx \wedge dy.$$

Or – (alternate explanation)

Let x, y be the standard basis of $\mathbb{C}^2$ and define

this is what $dx \wedge dy$ gives when evaluated on these pairs of vectors

$$\begin{cases} \omega(x,x) = 0 \\ \omega(x,y) = 1 \\ \omega(y,x) = -1 \\ \omega(y,y) = 0 \end{cases}$$

<u>Check</u>: $\omega(gv, gw) = \det g \; \omega(v,w) \quad \forall\, v, w \in \mathbb{C}^2$

so ω is invariant under $SL(2,\mathbb{C})$.

So: $V \cong V^*$ as a rep of $SL(2,\mathbb{C})$ and thus $SU(2)$.

What about $\overline{V}$?

(<u>Thm</u>: If we have a unitary rep of a grp, $V^* \cong \overline{V}$.)

So – as reps of $SU(2)$ $\overline{V} \cong V^* \cong V$.

But as reps of $SL(2,\mathbb{C})$ $\overline{V} \ncong V^* \cong V$.

The rep on V sends any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

to itself. i.e. a linear transformation of $V = \mathbb{C}^2$.

The rep on $\bar{V}$ sends the matrix to

$$SL(2,\mathbb{C}) \ni \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

(this is what the conjugate rep does - just conjugates each entry)

as a linear transf. on $\bar{V}$.

Saying $V \not\cong \bar{V}$ means there's no invertible 2x2 matrix T st

$$\underbrace{T \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}}_{\text{antiholomorphic}} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} T}_{\text{holomorphic funct}} \quad \forall\, a,b,c,d \in \mathbb{C} \text{ w/ } ad-bc=1$$

antiholomorphic: meaning complex conjugate of a holomorphic funct.

holomorphic funct (constant functs are holomorphic)

RHS has matrix entries holomorphic (analytic) in a,b,c,d;
LHS has entries antiholomorphic (complex conj. of holomorphic) in a,b,c,d. Only functs that are holo. & antiholom. are constants, so $T=0$.

So — we have two different "spinor reps" of the double cover of Lorentz group:

left-handed spinors: V

right-handed spinors: $\overline{V}$

(neutrinos / antineutrinos)

but these become equivalent when restricted to $SU(2)$, the double cover of the rotation group.

To get bigger reps & irreps of $SU(2)$ and $SL(2,\mathbb{C})$, we'll use a standard trick:

Suppose (ρ, V) is a rep. of G.

We can form the rep $(\rho^{\otimes n}, V^{\otimes n})$ where

$$\rho^{\otimes n}(g)\underbrace{(v_1 \otimes \cdots \otimes v_n)}_{V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}} = \rho(g)v_1 \otimes \cdots \otimes \rho(g)v_n$$

These are never irreducible (if $n > 1$) because we can define projection operators (square them, we get themself back):

$$p_S, p_A : V^{\otimes n} \longrightarrow V^{\otimes n}$$

P_S is "<u>symmetrization</u>":

$$P_S(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

and P_A is "<u>antisymmetrization</u>":

$$P_A(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

Ex) $n=2$

$$P_S(v \otimes w) = \underbrace{\tfrac{1}{2}(v \otimes w + w \otimes v)}_{= v \otimes v \text{ if } v=w} \Big/ \quad P_A(v \otimes w) = \underbrace{\tfrac{1}{2}(v \otimes w - w \otimes v)}_{= 0 \text{ if } v=w}$$

If we have n identical "bosons" (e.g. photons, mesons, etc) each of which has Hilbert space H, the collection of all the them has Hilbert space:

$$S^n H = \{ P_S \psi : \psi \in H^{\otimes n} \}$$

for "fermions" (e.g. protons, neutrons, electrons, quarks) we antisymmetrize and instead use:

$$\Lambda^n H = \{ P_A \psi : \psi \in H^{\otimes n} \}$$

(Note - can't put 2 electrons in the same state.)

Same works for a v. space V.

We'll look at $V = \mathbb{C}^2$ & $\overline{V}$ and get irreps of $SU(2)$ and $SL(2,\mathbb{C})$ by forming: $S^n V,\ \Lambda^n V$
$S^n \overline{V},\ \Lambda^n \overline{V}$