11/26/02

Suppose (ρ, V) is a rep of G on V. Define the "symmetrizer"

$$p_s : V^{\otimes n} \longrightarrow V^{\otimes n} \quad \text{by}$$

$$p_s(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!}\sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \qquad v_i \in V$$

with $p_s^2 = p_s$. (Check)

Check: p_s is an intertwiner:

$(\rho^{\otimes n}, V^{\otimes n})$ is a rep of G with

$$\rho^{\otimes n}(g)(v_1 \otimes \cdots \otimes v_n) = \rho(g)v_1 \otimes \cdots \otimes \rho(g)v_n.$$

and

$$p_s \, \rho^{\otimes n}(g) = \rho^{\otimes n}(g) \, p_s.$$

Either of these applied to $v_1 \otimes \cdots \otimes v_n$ gives:

$$\frac{1}{n!}\sum_{\sigma \in S_n} \rho(g) v_{\sigma(1)} \otimes \cdots \otimes \rho(g) v_{\sigma(n)}$$

Since p_s is an intertwiner,

$$S^n V = \text{range } p_s \subseteq V^{\otimes n}$$

is a subrep. of $V^{\otimes n}$.

We need to check:

$$x \in S^n V \Rightarrow \rho^{\otimes n}(g) x \in S^n V.$$

pf: $x \in S^n V \Rightarrow x \in \text{range } p_s$

$$\Rightarrow \exists\, y \in V^{\otimes n} \text{ st } x = p_s(y)$$

$$\Rightarrow \rho^{\otimes n}(g) x = \rho^{\otimes n}(g) p_s y$$

$$= p_s \rho^{\otimes n}(g) y$$

range p_s

$$\Rightarrow \rho^{\otimes n}(g) x \in S^n V.$$

Similarly – $\Lambda^n V = \text{range } p_A \subseteq V^{\otimes n}$

is also a subrep, since p_A is also an intertwiner.

Now let (ρ, V) be the defining rep of $G = SL(2, \mathbb{C})$ on $V = \mathbb{C}^2$. We get reps

$$\Big(\underbrace{\rho^{\otimes n}\Big|_{S^n V}}_{S^n \rho}, S^n V \Big) := (S^n \rho, S^n V)$$

on the symmetrized tensor powers of V. Since $SU(2, \mathbb{C}) \subseteq SL(2, \mathbb{C})$, we get a rep of $SU(2)$, and it's irreducible in both cases.

$(SL(2)$ - double cover of Lorentz grp$)$

These are irreducible, even when we restrict them to $SU(2) \subseteq SL(2, \mathbb{C})$. In fact, these are all the irreducible reps of $SU(2)$; but let's just show they are irreducible.

Note: If smaller space has no invariant subspace, then neither does larger space.

Thm: $(S^n \rho, S^n V)$ is an irrep. of $SU(2)$.

pf: $V = \mathbb{C}^2$ has a basis

$$\uparrow = \text{spin up} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \downarrow = \text{spin down} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$J_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \qquad J_z \uparrow = 1/2 \uparrow \quad \text{spin up state}$$

$$J_z \downarrow = -1/2 \downarrow \quad \text{spin down state}$$

$V^{\otimes n}$ has ~~a~~ basis of vectors like

$$\uparrow \otimes \downarrow \otimes \downarrow \otimes \uparrow \otimes \uparrow \otimes \downarrow \quad (n=6)$$

$S^n V$ has a basis of vectors like

$$p_s(\uparrow \otimes \downarrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \downarrow) =$$

$$p_s(\uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \downarrow \otimes \downarrow)$$

i.e. $$p_s(\underbrace{\uparrow \otimes \uparrow \otimes \dots \otimes \uparrow}_{n}) = \uparrow^n$$

$$p_s(\underbrace{\uparrow \otimes \uparrow \otimes \dots \otimes \uparrow}_{n-1} \otimes \downarrow) = \uparrow^{n-1} \downarrow$$

$$\vdots$$

$$p_s(\underbrace{\downarrow \otimes \downarrow \otimes \dots \otimes \downarrow}_{n}) = \downarrow^n$$

$\dim S^n V = n+1$

$S^n V = \{$ homogeneous degree-n polynomials in 2 variables $\}$

$S^n \mathbb{C}^k = \{$ homogeneous degree-n polynomials in k variables $\}$

Want to show these form an irrep of $SU(2)$.

Suppose $\mathcal{U} \subseteq S^n V$ is a subrepresentation:

$$S^n\rho(g) : \mathcal{U} \longrightarrow \mathcal{U} \quad \forall\, g \in SU(2)$$

Need to show $\mathcal{U} = \{0\}$ or $\mathcal{U} = S^n V$.
Assume $\mathcal{U} \neq \{0\}$. Let $0 \neq \psi \in \mathcal{U}$, and we'll show $\mathcal{U} = S^n V$.

Recall – if $X \in su(2)$, then $\exp(tX) \in SU(2)$, so –

$$S^n\rho\,(\exp(tX))\,\psi \in \mathcal{U}.$$

Differentiating w/r/t t:
(how to get a rep of Lie alg from rep of Lie grp)

$$\frac{d}{dt}\, S^n\rho\,(e^{tX})\,\psi \in \mathcal{U} \quad \forall\, t.$$

Recall – $\displaystyle \frac{d}{dt}\, S^n\rho\,(e^{tX})\Big|_{t=0} = dS^n\rho(X)$

and $dS^n\rho$ is a rep of $su(2)$ on $S^n V$.

So – $dS^n\rho(X)\,\psi \in \mathcal{U} \quad \forall\, X \in su(2)$.

Take X to be:

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$su(2) = \{ x : \mathbb{C}^2 \to \mathbb{C}^2 \mid tr(x) = 0,\ x^* = -x \}$

Note — J_+, J_- aren't in $su(2)$, they're in $sl(2)$.

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

in $sl(2)$ these are in $su(2)$.

J_+, J_- are linear combs. of complex multiples of matrices in $su(2)$.

* [In fact, every matrix in $sl(2,\mathbb{C})$ is of the form $x + iy$ where $x, y \in su(2)$.]

Thus, starting from our rep $dS^n\rho$ of $su(2)$ we can define a rep α of $sl(2,\mathbb{C})$ by:

$$\alpha(x+iy) = \alpha(x) + i\alpha(y)$$

$$:= dS^n\rho(x) + i\, dS^n\rho(y)$$

(replacing i by $-i$ we get conjugate rep)

Since $dS^n\rho(x)\psi \in \mathcal{U}$ $\forall x \in su(2)$ we also know

$$\alpha(x+iy)\psi \in \mathcal{U} \quad \forall x+iy \in sl(2,\mathbb{C})$$

(by defn of $\alpha(x+iy)$).

Write $\psi \in S^nV$ as:

$$\psi = a_0 \uparrow^n + a_1 \uparrow^{n-1}\downarrow + \ldots + a_n \downarrow^n$$

$$\mathcal{U} \ni \alpha(J_-)\psi = a_0 n \uparrow^{n-1}\downarrow + a_1(n-1)\uparrow^{n-2}\downarrow^2 + \ldots + a_{n-1}\downarrow^n + 0$$

takes one $\uparrow$ & turns it to $\downarrow$.

what it does to $\downarrow^n$.

Since $\mathcal{U}$ is an invariant subspace, we can do this again & end up in $\mathcal{U}$.

$$\mathcal{U} \ni [\alpha(J_-)]^2\psi = a_0 n(n-1)\uparrow^{n-2}\downarrow^2 + \ldots$$

Continuing doing this — we get

$\vdots$

$$\mathcal{U} \ni [\alpha(J_-)]^k\psi = (\text{some } \#)\downarrow^n$$

for some k, we get all spin down state.

We know $\downarrow^n$ is in $\mathcal{U}$.

We can do the same thing w/ J_+:

$$\alpha(J_+)\downarrow^n = n\uparrow\downarrow^{n-1}$$

$$\alpha(J_+)^2\downarrow^n = n(n-1)\uparrow^2\downarrow^{n-2}$$

$$\vdots$$

so we get a basis of $S^n V$ lying in $\mathcal{U}$, so

$\downarrow^n, \uparrow^n \in \mathcal{U} \Rightarrow \mathcal{U} = S^n V.$ ∎

Riemann sphere, heavenly sphere (space of directions we can look), same as states of spin-1/2 reps.

We call $(S^n\rho, S^n V)$ the "spin-j rep" of $SU(2)$ or of $SL(2,\mathbb{C})$ where $j = n/2$.

So (ρ, V) is called spin-1/2 representation.

V itself is closely related to the heavenly sphere: $\mathbb{C}P^1 = S^2$ is space of all 1-d subspaces of $V = \mathbb{C}^2$.

Let $L_x \subseteq \mathbb{C}^2$ be the 1-dim'l complex subspace corresp. to the point $x \in \mathbb{C}P^1$.

L_x is a 1-dim'l vector bundle (have a v.space $\forall$ pt)

$\mathbb{C}P^1$ - Riemann sphere

There's a 1-dim'l vector bundle L over $\mathbb{C}P^1$ w/ the L_x's as its fibers.

Given $f \in V^*$, it restricts to a linear functional on each $L_x \subseteq V$.

i.e. we get $f_x \in L_x^*$. So we get a section $\tilde{f}$ of the bundle L^* whose fibers are L_x^*.

(section - $\forall x$, we get an elt. of fiber L_x)

This is a holomorphic section of L^* (locally look like holomorphic functs) and in fact:

<u>Thm</u> - All holomorphic sections of L^* are of the form $\tilde{f}$ for some $f \in V^*$.

(only holomorphic functs on Riemann sphere are constants by Louiville.)

If $\Gamma(L^*) = \{$ holomorphic sections of $L^*\}$ we get

$$\Gamma(L^*) \cong V^* \quad \text{(restatement of above thm).}$$

$$\Gamma(L^*) \cong V^* \cong V$$

as reps of $SL(2,\mathbb{C})$.