12/3/02 <u>Standard</u> <u>Model</u>

Every irrep of $SU(2)$ is equivalent to the "spin-j rep",

$$S^n V \quad \text{where}$$

$V = \mathbb{C}^2$ and $j = \frac{n}{2}$.

There are also reps of $SL(2,\mathbb{C})$, which are also irreducible. We saw

$$V \cong \overline{V} \text{ as reps of } SU(2) \text{ but not } SL(2,\mathbb{C}).$$

So $\overline{V}$ is an irrep of $SL(2,\mathbb{C})$ not of the form $S^n V$, so there are other irreps of $SL(2,\mathbb{C})$.

$\boxed{\text{Thm:}}$ Every finite dim'l irrep of $SL(2,\mathbb{C})$ is equivalent to

$$S^n V \otimes S^m \overline{V},$$

called the "<u>spin-(j,k)</u>" irrep where $j = n/2$, $k = m/2$.

Let's now consider the Poincaré group

$$P = SO_0(3,1) \ltimes \mathbb{R}^4 \qquad (\mathbb{R}^4 \text{ normal subgrp of } P)$$

or really its double cover:

$$\tilde{P} = SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$$

Most of the unitary irreps of $\tilde{P}$ are infinite-dim'l and they consist of functions $\psi : \underset{\text{Minkowski spacetime}}{\mathbb{R}^4} \longrightarrow S^n V \otimes S^m \overline{V}$

$$\psi : \mathbb{R}^4 \longrightarrow S^n V \otimes S^m \overline{V}$$

satisfying some PDE which depends on (j,k) and also on a "mass," $m \geq 0$.

Examples that appear in Standard Model:

① Massive scalar field

Here $m \geq 0$ and $(j,k) = 0$ (so that $\psi : \mathbb{R}^4 \longrightarrow \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$) and the Hilbert space of our representation consists of functs

$$\psi : \mathbb{R}^4 \longrightarrow \mathbb{C} \qquad \begin{pmatrix} S^0 V \cong \mathbb{C}, \\ S^0 \overline{V} \cong \mathbb{C} \end{pmatrix}$$

satisfying the Klein-Gordon equation:

$$(\Box + m^2)\psi = 0 \quad \text{where}$$

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$ is called the D'Alembertian

($\Box$ does same thing for Mink. spacetime that Laplacian does)

If $m=0$ this describes waves moving in space w/ speed 1. (speed of light)

s.t. $\|\psi\| < \infty$ where $\|\psi\|^2 = \langle \psi, \psi \rangle$ using some $\langle \cdot, \cdot \rangle$.

$\tilde{P}$ acts on this space of functs. as follows:

First do $\tilde{P} \xrightarrow{2-1} P$ to get $g \in P$ and then

$$(\rho(g)\psi)(x) = \psi(g^{-1}x) \qquad x \in \mathbb{R}^4$$

rep ρ of Poincaré grp on space of functs

Check this gives a rep: i.e. $\rho(g)\rho(h) = \rho(gh)$ & $\rho(1) = 1$.

② <u>Massless</u> <u>left-handed</u> <u>spinor</u> <u>fields</u>

This rep. corresponds to $m = 0$ and $(j,k) = (1/2, 0)$ and consists of functs

$$\psi : \mathbb{R}^4 \longrightarrow \mathbb{C}^2 = V$$

$$\left(\begin{array}{l} S^1 V = V = \mathbb{C}^2 \\ S^0 \bar{V} = \mathbb{C} \quad \text{so} \quad S^1 V \otimes S^0 \bar{V} = \mathbb{C}^2 \otimes \mathbb{C} \cong \mathbb{C}^2 \end{array} \right)$$

We call elts of V "left-handed spinors" satisfying the <u>Dirac</u> <u>eqn</u>:

$$\not{\partial} \psi = 0$$

where
$$\not{\partial} = \sigma_0 \frac{\partial}{\partial t} + \sigma_1 \frac{\partial}{\partial x} + \sigma_2 \frac{\partial}{\partial y} + \sigma_3 \frac{\partial}{\partial z}$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are "Pauli matrices" – certain 2×2 matrices.

$\tilde{P}$ acts on the space of solns as follows:

given $\tilde{g} \in \tilde{P}$ form $g \in P$ and define

$$(\rho(g)\psi)(x) = (\tilde{h}\psi)(g^{-1}x) \quad \text{where}$$

$\tilde{h} \in SL(2,\mathbb{C})$ comes from $\tilde{g} \in \underbrace{SL(2,\mathbb{C})}_{\uparrow\ \tilde{h}} \ltimes \mathbb{R}^4$

anti-particles related to conjugate rep.

Again we really need a Hilbert space of ψ w/

$$\langle \psi, \psi \rangle < \infty$$

② ′ Maseless right-handed spinors

This corresponds to $m=0$ and $(j,k) = (0, 1/2)$ i.e.

$$\psi : \mathbb{R}^4 \longrightarrow \overline{V} \text{ satisfying the Dirac eqn.}$$

This is the conjugate rep of ② so it describes "antiparticles" of the particles described in ②.

③ · <u>Massless vectors</u>

This corresponds to $m=0$ and $(j,k) = (1/2, 1/2)$.

The $(1/2, 1/2)$ rep. of $SL(2,\mathbb{C})$ is on

$$S^1 V \otimes S^1 \bar{V} = V \otimes \bar{V} \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{C} \right\}$$

Recall 2×2 self-adjoint matrices, $\mathcal{H}$ (from begin. of course) is isomorphic to $\mathbb{R}^4$, i.e. Minkowski space. Complex lin. combs. of these give <u>all</u> 2×2 complex matrices.

So, $\quad \mathbb{C}\otimes\mathcal{H} \cong V\otimes\bar{V}$

We call $\mathbb{C}\otimes\mathcal{H} \cong V\otimes\bar{V}$ the "<u>vector" rep</u> of $SL(2,\mathbb{C})$.

Our "massless vector rep" of $\tilde{P}$ consists of functs

$$\psi : \mathbb{R}^4 \longrightarrow \underset{\substack{\uparrow \\ \text{4d complex v-space}}}{V\otimes\bar{V}} \cong \mathbb{C}\otimes\mathcal{H}$$

satisfying <u>Maxwell's eqns</u>.

These describe electromagnetic fields, aka light, aka photons.

For ① & ③ the rep its own conjugate so they describe particles that are their <u>own</u> antiparticle.

The (old) Standard Model. This describes all <u>known</u> particles and forces except gravity, i.e. 10% of what's out there.

To describe it, we need to specify

Standard model
- a group G
- certain irreps of G (kinds of particles)
- certain intertwiners (kinds of interactions)

$$G = \tilde{P} \times SU(3) \times \underbrace{SU(2) \times U(1)}$$

Poincaré grp

strong force – holds quarks together to form protons, neutrons

electroweak force – electromagnetism & weak force

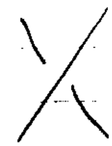
$n \longrightarrow p + e + \bar{\nu}_e$

neutrino

radioactive decay

The particles go like this:

Kind of ways particles can act

Bosons	Fermions
$j+k \in \mathbb{Z}$	$j+k \notin \mathbb{Z}$
 $v \otimes w \mapsto w \otimes v$	 $v \otimes w \mapsto -w \otimes v$

Scalars $(0,0)$, $m>0$	Vectors $(1/2, 1/2)$, $m=0$	Spinors $(1/2, 0)$ & $(0, 1/2)$: leptons (light-ones) (trivial reps of $SU(3)$)	quarks (nontrivial reps of $SU(3)$)
Higgs boson (unseen)	• 8 gluons (strong force)	• electron e &, neutrino ν_e	• down &, up
	• W^+, W^-, Z (weak force)	• muon μ (resembles electron but 200 times heavier & ν_μ neutrino	• strange &, charm
	• γ (photon) (electromagnetic force)	• tauon τ and ν_τ neutrino	• bottom &, top

Also – the antiparticles: scalars &, vectors are their own antiparticles

* spinors are not.

$\mathbb{C}$ stands for the trivial rep

12/5/02

$$G = \tilde{P} \times SU(3) \times SU(2) \times U(1)$$

<u>Bosons</u>

unaffected by strong force

Higgs boson [massive scalar] $\otimes \mathbb{C} \otimes \mathbb{C}^2 \otimes \rho_1$

Gauge bosons:

gluons [massless vector] $\otimes (\mathbb{C} \otimes su(3)) \otimes \mathbb{C} \otimes \mathbb{C}$

complexified $su(3)$

W's [massless vector] $\otimes \mathbb{C} \otimes (\mathbb{C} \otimes su(2)) \otimes \mathbb{C}$

X [massless vector] $\otimes \mathbb{C} \otimes \mathbb{C} \otimes (\mathbb{C} \otimes u(1))$

<u>Fermions</u>

Leptons:

1st generation

$\begin{pmatrix} \nu_e^L \\ e^L \end{pmatrix}$ [massless left-handed spinors] $\otimes \mathbb{C} \otimes \mathbb{C}^2 \otimes \rho_{-1}$ ← comes from work...

neutrino of electron & electron — but the left-handed ones

e^R [massless right-handed spinors] $\otimes \mathbb{C} \otimes \mathbb{C} \otimes \rho_{-2}$

2nd generation

$\begin{pmatrix} \nu_\mu^L \\ \mu^L \end{pmatrix}$ [ ] $\otimes \mathbb{C} \otimes \mathbb{C}^2 \otimes \rho_{-1}$

μ^R [ ] $\otimes \mathbb{C} \otimes \mathbb{C} \otimes \rho_{-2}$

mu neutrinos, muon

3rd generation

$$\begin{pmatrix} \nu_\tau^L \\ \tau^L \end{pmatrix} \quad [\qquad\qquad] \otimes \mathbb{C} \otimes \mathbb{C}^2 \otimes \rho_{-1}$$

$$\tau^R \quad [\qquad\qquad] \otimes \mathbb{C} \otimes \mathbb{C} \otimes \rho_{-2}$$

Quarks:

1st generation (up/down)

$$\begin{pmatrix} u^L \\ d^L \end{pmatrix} \quad [\qquad\qquad] \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \rho_{1/3}$$

$$u^R \quad [\qquad\qquad] \otimes \mathbb{C}^3 \otimes \mathbb{C} \otimes \rho_{4/3}$$

$$d^R \quad [\qquad\qquad] \otimes \mathbb{C}^3 \otimes \mathbb{C} \otimes \rho_{-2/3}$$

2nd generation (charm, strange)

$$\begin{pmatrix} c^L \\ s^L \end{pmatrix}, \quad c^R, \quad s^R$$

same as above

3rd gen. (top, bottom)

$$\begin{pmatrix} t^L \\ b^L \end{pmatrix}, \quad t^R, \quad b^R$$

We have lots of intertwiners describing interactions!

In the new standard model we have right-handed neutrinos:

$$\begin{pmatrix} \nu_e^L \\ e^L \end{pmatrix} \quad [\text{massless spinor}] \otimes \mathbb{C} \otimes \mathbb{C}^2 \otimes \rho_{-1}$$

$$\nu_e^R \quad [\qquad " \qquad] \otimes \mathbb{C} \otimes \mathbb{C} \otimes \rho_0$$

$$e^R \quad [\qquad " \qquad] \otimes \mathbb{C} \otimes \mathbb{C} \otimes \rho_{-2}$$

Remarks on chart

Recall — $SU(3)$ is the strong nuclear force

$SU(2) \times U(1)$ — electroweak

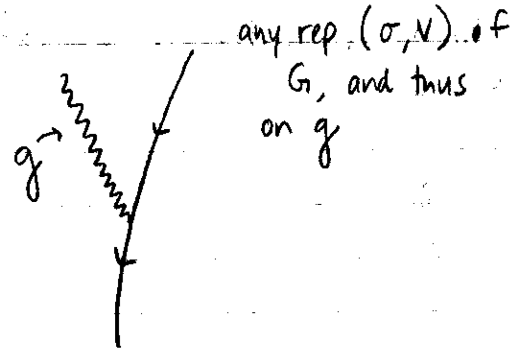
- every Lie grp G has a 'god-given' rep on its Lie alg $\mathfrak{g}$ called the <u>adjoint rep</u>.

If G is a matrix Lie group, $\mathfrak{g}$ consists of matrices too, and the adjoint rep ρ of G on $\mathfrak{g}$ is:

$$\rho(g)v = \underbrace{gvg^{-1}}_{\in \mathfrak{g}}, \qquad g \in G,\ v \in \mathfrak{g}$$

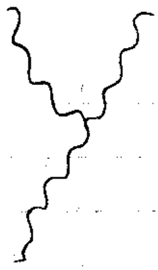
- gluons are elts. carrying the strong force $SU(3)$ (Lie alg elts act on reps of the Lie alg)

- any rep of G can become a rep of $\mathfrak{g}$, and Lie alg elts act on it.



$$\mathfrak{g} \otimes V \to V \qquad x \otimes v \mapsto d\sigma(x)v \qquad x \in \mathfrak{g},\ v \in V$$

This Feynman diagram describes how gauge bosons assoc. to the group G act on particles e.g. if $(\sigma, V) = (\rho, \mathfrak{g})$ we get



8 gluons — strong force — $SU(3)$

3 W's — electroweak force — $SU(2)$

1 X — — $U(1)$

In nature we see not 3 W's and an X, instead we see W^+, W^-, Z° and γ (photon)

W_1, W_2, W_3 corresp. to a basis of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.

X corresp. to a basis of $\mathfrak{u}(1)$

So they form a basis of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ but $W^\pm$, Z°, γ form a different basis.

$$\gamma = W_3 + \frac{X}{2}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

electric charge = $\text{isospin}_3 + \dfrac{\text{hypercharge}}{2}$

Recall — our old notation for reps of $U(1)$ was

$$\rho_k(e^{i\theta}) = e^{ik\theta} \qquad k \in \mathbb{Z}$$

k was "charge" if this $U(1)$ was for electromagnetism.
For hypercharge people often normalize things differently.

Fermions in Standard Model:
come in 2 classes:

leptons - don't interact w/ strong force
(transform in trivial rep of $SU(3)$)

quarks - do interact w/ strong force
(transform in defining rep of $SU(3)$ on $\mathbb{C}^3$)

electric charge = isospin_3 + $\dfrac{\text{hyperchange}}{2}$

for neutrino^L

$$0 = \tfrac{1}{2} + -\tfrac{1}{2}$$

for electron^L $\quad -1 = -\tfrac{1}{2} + -\tfrac{1}{2}$

$\begin{pmatrix} \nu_e^L \\ e^L \end{pmatrix}$ means $\mathbb{C}^2$ has basis $\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{corr. to } \nu_e^L}$ & $\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{correp to } e^L}$

$$W_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W_3\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}$$ so isospin$_3$ of ν_e^L is $1/2$

$$W_3\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$$ so isospin$_3$ of e^L is $-1/2$

for right handed electron:

e^R:

$$\text{electric charge} = \text{isospin}_3 + \frac{\text{hypercharge}}{2}$$

$$-1 = \underbrace{0}_{\text{because } e^R \text{ is in the trivial rep of } su(2)} + \left(-\frac{2}{2}\right)$$

because e^R is in the <u>trivial</u> rep of $su(2)$

B. Franklin incorrectly assigned a sign to charge. This is why the electron is negative.

$$\text{charge} = \text{isospin}_3 + \frac{\text{hyperchange}}{2}$$

u^L	$\frac{2}{3}$	$=$	$\frac{1}{2}$	$+$	$^{1/3}/_2$
d^L	$-\frac{1}{3}$	$=$	$-\frac{1}{2}$	$+$	$^{1/3}/2$
u^R	$\frac{2}{3}$	$=$	0	$+$	$^{4/3}/_2$
d^R	$-\frac{1}{3}$	$=$	0	$+$	$^{(-2/3)}/_2$

look at 3rd entry in tensor product

Now we're forced into using a new convention for our reps!

$$\rho_K(e^{i\theta}) = e^{3i\theta K} \qquad K \in \mathbb{Z}/3$$