Let's summarize what we've done so far:

We have a space with basis $\psi_0, \psi_1, \psi_2, \ldots$ on which various operators act:

- **Creation**: $a^* \psi_n = \psi_{n+1}$
- **Annihilation**: $a \psi_n = n \psi_{n-1}$
- **Position**: $q = \frac{a + a^*}{\sqrt{2}}$
- **Momentum**: $p = \frac{a - a^*}{\sqrt{2} i}$

**Hamiltonian**: $H = \frac{1}{2}(p^2 + q^2)$, $H \psi_n = (n + \frac{1}{2}) \psi_n$

**Renormalized Hamiltonian** (subtracting off ground state energy):
- $\hat{N} = H - \frac{1}{2}$
- $\hat{N} \psi_n = n \psi_n$

**Fourier Transform**:
- $F = (-it)^N$
- $F \psi_n = (-it)^n \psi_n$
- $= e^{-itN}$ where $t = \frac{\pi}{4} - \frac{1}{4}$ a quarter period.

In terms of the Fock representation, we call this vector space $\mathbb{C}[z]$. We call $\psi_n$ "$z^n$". Thus we get:

- $a^* = M_2^z$ (multiplication by $z$)
- $a = \frac{d}{dz}$
- $N = M_z \frac{d}{dz}$

$(Nz^n = z \frac{dz^n}{dz} = n z^n)$
To evolve a state $\phi \in \mathbb{C}[z]$ by time $t$ we apply $e^{-itN}$.

What does this do?

$$\left( e^{-itN} \phi \right)(z) = \sum_{n=0}^{\infty} a_n e^{-itN} z^n$$

$$= \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} a_n (e^{-it} z)^n$$

$$= \phi(e^{-it} z)$$

So: if we think of $z \in \mathbb{C}$ as a point in phase space

we can think of our state as a function of position & momentum and evolve it in time by the rule $z \mapsto e^{-it} z$, same as the classical rule:

$$q(t) = \cos t \ q + \sin t \ p$$
$$p(t) = -\sin t \ q + \cos t \ p$$

$\iff z(t) = e^{-it} z$

Moral: the Fourier transform is a quarter-turn in phase space (in Fock rep.)

So much for the quantum harmonic oscillator...
Now: CATEGORIFY IT ALL!

Whenever we see "n ∈ N" replace it with "n, the n-element set."

First some notes about the article "the" in the above:

- Not all n-element sets are equal (unless n = 0)
- All n-element sets are isomorphic
- Not all n-elt. sets are isomorphic in a unique way (unless n = 0, 1)

So we must use "the" carefully.

Now let's categorify polynomials, or more generally formal power series, like

\[ \phi(z) = \sum a_n z^n \quad \text{where } a_n \in \mathbb{N} \]

We'll say a "structure type" or "species" is a type of structure that you can put on a finite set. (Not the real definition yet!)

Given a structure type \( \Phi \) let \( \Phi_n \in \mathbb{N} \) be the number of ways you can put this structure on an n-element set. Then define the generating function of \( \Phi \), say \( \Phi(z) \), to be

\[ \Phi(z) = \sum \frac{\Phi_n z^n}{n!} \]

Examples:

1) \( \Phi = "2-colorings" \) - ways of coloring each element of a finite set black or white

- \( \Phi_n = 2^n \)
So \(|\Psi|_n = \sum_{n=0} \frac{2^n z^n}{n!} = e^{2z}\)

We think of \(e^{2z}\) as the decategorified version of "2-colorings".

2) \(\Psi = \text{"k-colorings"} = \text{ways of mapping our finite set to } k\)
   = \text{ways of coloring our finite set with}
   \text{the "colors" } \{1, 2, 3, \ldots, k\}\)

- A 3-coloring of 6

\(\Psi_n = \# \text{ ways of } k\text{-coloring an } n\text{-elt set} = k^n\)

So \(|\Psi|_n = \sum_{n=0} \frac{k^n z^n}{n!} = e^{kz}\)

If \(k=0\) we get \(e^{0z} = 1 = \sum \frac{a_0 z^n}{n!}\) where \(a_n = 0\) unless \(n=0\), in which case \(a_0 = 1\).

If \(k=1\) we get \(e^{z} = \sum \frac{1^n z^n}{n!} = e^z\).

3) Suppose \(\Psi\) & \(\Psi\) are structure types.
   Let's invent a structure type \(\Psi + \Psi\) such that
   \(|\Psi + \Psi| = |\Psi| + |\Psi|\).

\(|\Psi + \Psi| = \sum (\Psi + \Psi)_n \frac{z^n}{n!}\)

so we need
\((\Psi + \Psi)_n = \Psi_n + \Psi_n\)
The answer: A \((\Phi + \Psi)\)-structure on an \(n\)-elt. set is:

a \(\Phi\)-structure \(\lor\) a \(\Psi\)-structure

really a disjoint union, since a \(\Phi\)-structure might also be a \(\Psi\)-structure.

E.g. If \(\Phi = "2\text{-colorings}"\)
\(\Psi = "3\text{-colorings}"\)

then \(\Phi + \Psi = "2\text{-colorings} \lor 3\text{-colorings}"\)

\(|\Phi + \Psi| (z) = e^{2z} + e^{3z} = \sum_{n=0}^{\infty} \frac{2^n + 3^n}{n!} z^n\)

4) Next puzzle: find \(\Phi \Psi\) s.t. \(|\Phi \Psi| = |\Phi||\Psi|\)

Want

\(|\Phi \Psi| (z) = |\Phi| (z) |\Psi| (z)\)

\[= \sum_{n} \frac{\Phi_n z^n}{n!} \sum_{m} \frac{\Psi_m z^m}{m!}\]

\[= \sum_{n,m} \frac{\Phi_n \Psi_m}{n! m!} z^{n+m}\]

\[= \sum_{p} \sum_{n,m: n+m=p} \frac{\Phi_n \Psi_m}{n! m!} z^p \frac{z^p}{p!}\]

\[= \sum_{p} \sum_{0 \leq n \leq p} \binom{p}{n} \Phi_n \Psi_{p-n} \frac{z^p}{p!}\]

\((\Phi \Psi)_p = \text{number of ways to chop up} p \text{ into 2 parts and put a } \Phi\text{-str. on first part and a } \Psi\text{-str. on 2nd part.}\)

Next time ... the Catalan Numbers.
Kansas University

6 November 2003

"CATALAN NUMBERS"

Named after Eugène Catalan since discovered by Gauss.

A magma is a set $M$ equipped w. a binary operation $\cdot : M \times M \rightarrow M$. Consider the free magma on one element $x$. The elements include:

$\begin{align*}
c_1 &= 1 & x \\
c_2 &= 1 & xx \\
c_3 &= 2 & (xx)x & x(xx) \\
c_4 &= 5 & (xx)(xx) & x((xx)x) & ((xx)x)x & x(x(xx)) & (x(xx))x & \vdots \\
\end{align*}$

We define $c_n$ to be the number of elts. built from $n$ $x$'s in the free magma on $x$. (Normally people use some different convention - but this seems more basic)

\[ a \quad b \]

\[ \begin{array}{c}
\text{This is a picture of } M \times M \rightarrow M \\
\text{(multiplication in our magma)}
\end{array} \]

This lets us draw elts of the free magma on $x$ as binary planar trees:

\[ x^2 \]
\[ x^2 \]
\[ x^2 \]
\[ ((xx)x)(x(xx)) \]

"a binary planar tree with 6 leaves"
And conversely:

\[((xx)x)((xx)(x(xx)))\]

\[
\downarrow
\]

So: \(c_n\) is the number of binary planar trees with \(n\) leaves.

There are other things the Catalan #s are good for.

Let's count paths on an \(nxn\) grid from SW corner to NE corner that only go N or E & never enter the shaded NW region.
Claim: The number of these paths is \( C_n \).

An element of the free magma on \( x \) with \( n \) \( x \)'s is secretly just a reverse Polish notation expression with \( n \) \( x \)'s an \((n-1)!\) "dots," which is secretly just a path of the above sort.

Next: Consider chopping a regular polygon into triangles:

Claim: There are \( C_n \) ways to chop a regular \((n+1)\)-gon into triangles.
How does this work:

Pick a side of the \((n+1)\)-gon to be the "output" side and then draw the tree that is Poincaré dual to the triangulation.

Conversely:

Let's calculate \(c_n\)!

Let \(T\) be the structure type "planar binary trees": a \(T\)-structure on a finite set is a way of making its elts into the leaves of a planar binary tree.

E.g. if our set is \(3 = \{0, 1, 2, 3\}\), a \(T\)-structure on it could be

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
Y & Y & Y & Y
\end{array}
\]

... etc.

for a total of \(3! \cdot c_3 = 12\) \(T\)-structures.
So let $T_n = n! \cdot c_n$ be the number of $T$-structures on the set $n$, & get the generating function:

$$|T| (z) = \sum_{n=0}^{\infty} \frac{T_n z^n}{n!} = \sum_{n=0}^{\infty} c_n z^n$$

$$= x + x^2 + 2x^3 + 5x^4 + \ldots$$

Recall:

1) Given str. types $\Psi$ & $\Psi'$, a "$\Psi + \Psi'$-str." on a set is a $\Psi$-structure xor $\Psi'$-str. on that set.

$$|\Psi + \Psi'| = |\Psi| + |\Psi'|$$

2) Given str. types $\Psi$ & $\Psi'$, to put a "$\Psi \Psi'$-str." on a set is to chop the set into 2 disjoint subsets & put a $\Psi$-str. on the first, a $\Psi'$-str. on the second.

3) There's a str. type $Z$ with $|Z| = z$.

Since the coefficient of $z^n$ in this power series is 0 unless $n=1$, in which case it's 1, there are no ways to put a $Z$ structure on a set unless it has 1 element, in which case there is one way. So we say

$Z$ = "being a 1-element set"
More generally, there's a structure type called \( \frac{Z^n}{n!} \), or "being an \( n \)-element set," & whose generating function is:

\[
\left| \frac{Z^n}{n!} \right| = \frac{z^n}{n!}
\]

If \( n = 0 \) we get a str. type 1 with

\[
1 = 1
\]

& is "being the empty set."

Use these to calculate \( |T|(z) \)!