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Quantum Gravity Seminar
HW #1 — *Groups as Categories*

In what follows, by **group** I always mean a group in the categorical sense: A group is a groupoid with one object. I use the traditional backward order of composition: fg means first g then f.

1. Let G and H be groups. A functor $F\colon G \to H$ is boring as an object map, so the only essential property of F is that $F(fg) = F(f)F(g)$ for any morphisms $f, g \in \hom_G(*, *)$. (The other property, preservation of identities, is easily derived from this one in the case of groups. This would not be the case if G and H were mere monoids.) Such a functor is usually called a **group homomorphism**.

2. Let $F, F'\colon G \to H$ functors between groups. Since G has only one object, $*$, and H has only one object, $\bullet$, a natural transformation is just a morphism $\alpha_*\colon \bullet \to \bullet$ in H, such that for any morphism g in G, we have $F'(g)\alpha_* = \alpha_* F(g)$.

3. Let $1_G\colon G \to G$ be the identity functor on the group G, and $\alpha\colon 1_G \Rightarrow 1_G$ a natural transformation. This is just a morphism α in G such that for any morphism g in G we have $g\alpha = \alpha g$. The set of all such natural transformations is usually called the **center** of G.

4. If G is a group with object $*$, a functor $F\colon G \to \mathrm{Vect}$ amounts to a choice of a vector space $V = F(*)$ and for each morphism g in G a linear transformation $F(g)\colon V \to V$, such that $F(gh) = F(g)F(h)$ for any two morphisms $f, g\colon * \to *$, and $F(1_*) = 1_V$. Better yet, we can consider a functor $F\colon G \to \mathrm{Vect}_0$. The groupoid Vect_0 has a full subcategory $\mathrm{GL}(V)$ whose only object is $V = F(*)$, so we can consider F to be a group homomorphism $F\colon G \to \mathrm{GL}(V)$ — in other words, a **representation** of G on V.

5. Suppose $F, F'\colon G \to \mathrm{Vect}$ are representations of a group G with object $*$. If we write $V = F(*)$ and $V' = F'(*)$, a natural transformation $\alpha\colon F \Rightarrow F$ is a linear map $\alpha_*\colon V \to V'$ such that $F'(f)\alpha_* = \alpha_* F(f)$. So, such a natural transformation is just an **intertwiner**.