Groups as categories

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1. *Homomorphisms as functors.*

A functor $F\colon G \to H$ is such that $F(g\colon x \to y)\colon F(x) \to F(y)$, compatible with composition

$$F(g_1\colon x \to y) \circ F(g_2\colon y \to z) = F(g_1 \circ g_2\colon x \to z)\colon F(x) \to F(z)$$

If G, H are groupoids with one object,

$$F(g_1) \circ F(g_2) = F(g_1 \circ g_2)$$

which means F is a homomorphism from the morphism group of G to that of H. Conversely, any group homomorphism defines a functor.

2. *Conjugation as a natural transformation.*

A natural transformation α between functors $F_1, F_2\colon G \to H$ assigns an endomorphism $\alpha_x\colon F_1(x) \to F_2(x)$ to each object x of G in such a way that that, for every $g\colon x \to y$ in G, $\alpha_x F_2(g) = F_1(g)\alpha_y$ in H.

If G and H are groupoids with a single object, a natural transformation between group homomorphisms F and G is a single group element $\alpha \in H$ such that

$$\alpha F_2(g) = F_1(g)\alpha \qquad \text{for all} \quad g \in G,$$

that is, $F_1(g)$ is the result of conjugating $F_2(g)$ by $\alpha \in H$.

3. *The center as the natural automorphisms of the unit.*

If, now, $F_1 = F_2 = 1_G\colon G \to G$, a natural transformation $\alpha\colon F_1 \Rightarrow F_2$ is a group element $\alpha \in G$ satisfying

$$\alpha g = g\alpha \qquad \text{for all} \quad g \in G,$$

that is, α is in the center of the group G.

4. *Group representations as functors.*

If G is a groupoid with one object, a functor $F\colon F \to \mathrm{Vect}$ is a choice of a vector space $V = F(\bullet)$ and a group homomorphism $F\colon G \to \mathrm{End(V)}$. In other words, F is a representation of the group G.

5. *Intertwiners as natural transformations.*

Suppose that F, F' are two representations of the group G. Then, a natural transformation $\alpha\colon F \Rightarrow F'$ is a linear map $\alpha\colon V \to V'$ such that

$$\alpha F'(g) = F(g)\alpha \qquad \text{for all} \quad g \in G.$$

The map α is called an intertwining operator between the representations F and F'.