Connections As Functors

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1. We have a smooth function $A : [a, b] \to \mathrm{End}(\mathbb{R}^n)$, then the theorem which asserts the existence and uniqueness of solutions to the initial value problem $\partial_t \psi(t) = A(t)\psi(t); \psi(t_0) = \psi_0$ gives us, for any initial condition $\psi(t_0) = \psi_0 \in \mathbb{R}^n$ for $t_0 \in [a, b]$, an element in $\mathbb{R}^n$ for every point in $[a, b]$, namely the value of the unique solution ψ to the D.E. at that point. This gives a unique map from $\mathbb{R}^n$ to $\mathbb{R}^n$ given any two points. Note that this map is linear since the derivative is, hence the effect of the map $F(f)$ on any basis of $\mathbb{R}^n$ determines the effect on the whole space, since a linear combination of solutions to IVP's gives a solution to the IVP with same linear combination of the initial values.

   Now to see that the hopeful monster F is, in fact, a functor, first since the maps $F(f)$ are linear (i.e. belong to $\mathrm{End}(\mathbb{R}^n) = \hom(\mathbb{R}^n, \mathbb{R}^n)$ in Vect), and second since the fact that $F(fg) = F(f)F(g)$ reduces to the linearity of the integral under the union of disjoint intervals (and the fact that integrating backwards gives inverses). Similarly, the property that $F(\mathrm{Id}) = \mathbb{1}_{\mathbb{R}^n}$ is just the fact that the integral over an interval of zero length is zero, hence the map we get takes every ψ_0 to itself by definition. Thus, F is a functor. It is unique because the maps are given by unique solutions to the IVP's.

2. First, notice that if any such map exists, it must be unique: any path $\gamma \in \mathcal{P}(X)$ gives, for every point of $[0, T]$ a map in $\mathrm{End}(\mathbb{R}^n)$, namely $A_{\gamma(t)}\gamma'(t)$, so this gives a differential equation on $[0, T]$ and by (1) there is a a unique map to $\mathrm{End}(\mathbb{R}^n)$ satisfying this differential equation on any interval on which the path is smooth (and therefore on the whole path, by composition). Thus, any path in X gives a morphism in Vect taking $\psi_0 \mapsto \psi(T)$ for the ψ solving the DE with initial value ψ_0.

   To see that the F which accomplishes this is a functor, we note that, as in (1), we automatically have $F(\mathrm{Id}) = \mathbb{1}_{\mathbb{R}^n}$ automatically, and functiorial properties of composition and inverses are again by properties of integrals (which work on paths just as on intervals).