Connections as Functors

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In this homework we'll see how a vector bundle E equipped with a connection over a manifold X gives a functor

$$F\colon \mathcal{P}(X) \to \mathrm{Vect}$$

where $\mathcal{P}(X)$ is the category of paths in X, defined in the last homework. Recall that objects of $\mathcal{P}(X)$ are points of X, while morphisms are piecewise-smooth paths in X. The functor F maps each point $x \in X$ to the the fiber of E over the point x. Similarly, F maps each path $\gamma\colon x \to y$ in X to a linear operator

$$F(\gamma)\colon F(x) \to F(y)$$

defined using parallel transport along the path γ.

To warm up, let's see how any linear ordinary differential equation gives a functor. I'll let you use this fact:

Theorem 1. *Let $\mathrm{End}(\mathbb{R}^n)$ be the algebra of linear operators from $\mathbb{R}^n$ to itself. Suppose $A\colon [a,b] \to \mathrm{End}(\mathbb{R}^n)$ is any smooth function and $t_0 \in [a,b]$. Given any vector $\psi_0 \in V$, the differential equation*

$$\frac{d\psi(t)}{dt} = A(t)\psi(t) \tag{1}$$

has a unique smooth solution $\psi\colon [a,b] \to \mathbb{R}^n$ with $\psi(t_0) = \psi_0$.

Sketch of Proof. *With the given initial conditions, Equation (1) is equivalent to the integral equation*

$$\psi(t) = \psi_0 + \int_{t_0}^{t} A(s)\psi(s)ds.$$

A solution of this equation is none other than a fixed point of the map T sending the function ψ to the function $T\psi$ given by

$$(T\psi)(t) = \psi_0 + \int_{t_0}^{t} A(s)\psi(s)ds.$$

T maps the Banach space of continuous $\mathbb{R}^n$-valued functions on $[a,b]$ to itself. If $\int_a^b \|A(s)\|ds = M$ then

$$\|T(\psi_1) - T(\psi_2)\| \le M\|\psi_1 - \psi_2\|.$$

We call a map with this property a **contraction** *if $M < 1$. An easy argument shows that any contraction on a Banach space has a unique fixed point, so our equation has a unique solution. If $M \not< 1$, we can chop the interval $[a,b]$ into smaller intervals for which this bound does hold, and prove the theorem one piece at a time.* □

1. Let $K[a,b]$ be the category whose objects are points of the interval $[a,b]$, with exactly one morphism from any object to any other. Given a function $A\colon [a,b] \to \mathrm{End}(\mathbb{R}^n)$ satisfying the conditions of Theorem 1, use the theorem to prove there is a unique functor

$$F\colon K[a,b] \to \mathrm{Vect}$$

such that:

- F sends any object to $\mathbb{R}^n$.
- F sends any morphism $f\colon t_0 \to t_1$ to the linear operator

  $$\psi_0 \mapsto \psi(t_1)$$

  where $\psi\colon [a,b] \to \mathbb{R}^n$ is the unique solution of Equation (1) with $\psi(t_0) = \psi_0$.

F is uniquely defined by Theorem 1. (In fact, I can say more explicitly that

$$F(f) = \mathcal{P}\exp \int_{t_0}^{t_1} A(t)\,\mathrm{d}t$$

where $\mathcal{P}\exp$ is the path-ordered exponential; this is because

$$\psi(t) = \mathcal{P}\mathrm{e}^{\int_{t_0}^{t} A(t')\,\mathrm{d}t'}\psi_0$$

is a solution to Equation (1) with $\psi(0) = \psi_0$. Just differentiate it to check.) The only question is whether this F is a functor.

First, if $f\colon t_0 \to t_1$ in $K[a,b]$, then $F(f)\colon \mathbb{R}^n \to \mathbb{R}^n = F(t_0) \to F(t_1)$ in Vect, *so F takes values of the proper types.*

Next, if $f\colon t_0 \to t_0$ is an identity in $K[a,b]$, then $F(f)\psi_0 = \psi(t_0) = \psi_0$, so that $F(f)$ is an identity in Vect. *(In terms of the exponential formula for F, $F(f) = \mathcal{P}\exp\int_0^0 A(t)\,\mathrm{d}t = \mathcal{P}\exp 0 = 1$.)*

Finally, if $f_1\colon t_0 \to t_1$ and $f_2\colon t_1 \to t_2$, then $F(f_1 f_2)\psi_0 = \psi(t_2)$ where ψ is the solution of Equation (1) with $\psi(t_0) = \psi_0$. Meanwhile, $F(f_1)\psi_0 = \psi(t_1)$ for the same ψ; let ψ_1 be this $\psi(t_1)$. Now notice that ψ is also the solution of Equation (1) with $\psi(t_1) = \psi_1$. Thus, $F(f_2)\psi_1 = \psi(t_2)$. Summing up,

$$F(f_1 f_2)\psi_0 = \psi(t_2) = F(f_2)\psi_1 = F(f_2)F(f_1)\psi_0.$$

In other words, $F(f_1 f_2) = F(f_2)F(f_1)$. (In terms of the exponential formula for F,

$$F(f_1 f_2) = \mathcal{P}\mathrm{e}^{\int_{t_0}^{t_2} A(t)\,\mathrm{d}t} = \mathcal{P}\mathrm{e}^{\int_{t_1}^{t_2} A(t)\,\mathrm{d}t + \int_{t_0}^{t_1} A(t)\,\mathrm{d}t} = \mathcal{P}\mathrm{e}^{\int_{t_1}^{t_2} A(t)\,\mathrm{d}t}\mathcal{P}\mathrm{e}^{\int_{t_0}^{t_1} A(t)\,\mathrm{d}t} = F(f_2)F(f_1).$$

Note that the equation $\mathrm{e}^{A_2 + A_1} = \mathrm{e}^{A_2}\mathrm{e}^{A_1}$ is not *always valid for $A_1, A_2 \in \mathrm{End}(\mathbb{R}^n)$; this is why we need path-ordered exponentials instead of plain old exponentials in this problem.)*

Therefore, F is indeed a functor.

More poetically, $K[a,b]$ is the category whose objects are **moments of time** *between time a and time b. The morphisms in this category are* **passages of time**. *Applied to the passage of time from t_0 to t_1, the functor F gives the* **time evolution operator** *mapping $\psi(t_0)$ to $\psi(t_1)$, where ψ is any solution of*

$$\frac{d\psi(t)}{dt} = A(t)\psi(t).$$

Next, suppose X is a smooth manifold and A is a smooth $\mathrm{End}(\mathbb{R}^n)$*-valued 1-form on X. For each point $x \in X$, such a thing gives a linear map*

$$A_x\colon T_x X \to \mathrm{End}(\mathbb{R}^n),$$

and A_x varies smoothly as a function of x. If we take $n = 1$, A becomes an ordinary 1-form and the following result reduces to a problem in the last homework assignment:

2. Suppose A is a smooth $\mathrm{End}(\mathbb{R}^n)$-valued 1-form on the manifold X. Show that there is a unique functor

$$F: \mathcal{P}(X) \to \mathrm{Vect}$$

such that

- F sends any point of X to $\mathbb{R}^n$.
- F sends any piecewise-smooth path $\gamma: [0,T] \to X$ to the linear operator

$$\psi_0 \mapsto \psi(T)$$

  where $\psi: [0,T] \to \mathbb{R}^n$ is the unique solution of the equation

$$\frac{d\psi(t)}{dt} = A_{\gamma(t)}(\gamma'(t))\,\psi(t) \tag{2}$$

  with $\psi(0) = \psi_0$.

Note that for a given γ, Equation (2) is simply Equation (1) with $[a,b]$ replaced by $[0,T]$ and $A(t)$ replaced by $A_{\gamma(t)}(\gamma'(t))$. Thus, $F(\gamma)$ is uniquely defined, by Theorem 1. (Again, there is a formula:

$$F(\gamma) = \mathcal{P}\exp\int_0^T A_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t = \mathcal{P}\exp\int_\gamma A.$$

Thus, the $n=1$ case is related to Exercise 1 in the previous homework assignment through the group homomorphism $\exp: \mathbb{R} \to \mathbb{R}^\times = \mathrm{End}(\mathbb{R}^1) \to \mathrm{Aut}(\mathbb{R}^1)$*.) In the previous assignment we did not need path-ordered exponentials, because* $\mathrm{End}(\mathbb{R}^1)$ *is commutative.*

Now, if $\gamma: x \to y$ in $\mathcal{P}(X)$, then $F(\gamma): \mathbb{R}^n \to \mathbb{R}^n = F(x) \to F(y)$ in Vect*, so again F takes values of the proper types.*

Next, if $\gamma: x \to x$ is the identity path on x, then $F(\gamma)\psi_0 = \psi(T) = \psi(0) = \psi_0$, so that $F(\gamma)$ is the identity operator on R^n. (In terms of the exponential formula for F, $F(1_x) = \exp\int_0^0 \mathrm{whatever}\,\mathrm{d}t = \exp 0 = 1$.)

Finally, if $\gamma_1: x \to y$ and $\gamma_2: y \to z$, then $F(\gamma_1\gamma_2)\psi_0 = \psi(T_1+T_2)$, where ψ is the solution of Equation (2) with $\psi(0) = \psi_0$. Meanwhile, $F(\gamma_1)\psi_0 = \psi(T_1)$ for this same ψ; let ψ_1 be this $\psi(T_1)$. If I translate ψ by T_1, to get $\phi(t) = \psi(T_1+t)$, then ϕ is the solution to Equation (2) with $\phi(0) = \psi(T_1) = \psi_1$. Thus, $F(\gamma_2)\psi_1 = \phi(T_2)$. Summing up,

$$F(\gamma_1\gamma_2)\psi_0 = \psi(T_1+T_2) = \phi(T_2) = F(\gamma_2)\psi_1 = F(\gamma_2)F(\gamma_1)\psi_0.$$

In other words, $F(\gamma_1\gamma_2) = F(\gamma_2)F(\gamma_1)$. (In terms of the exponential formula for F,

$$F(\gamma_1\gamma_2) = \mathcal{P}\mathrm{e}^{\int_{\gamma_1\gamma_2} A} = \mathcal{P}\mathrm{e}^{\int_{\gamma_2} A + \int_{\gamma_1} A} = \mathcal{P}\mathrm{e}^{\int_{\gamma_2} A}\mathcal{P}\mathrm{e}^{\int_{\gamma_1} A} = F(\gamma_2)F(\gamma_1).$$

Again, the equation $\mathrm{e}^{A_2+A_1} = \mathrm{e}^{A_2}\mathrm{e}^{A_1}$ is not valid for $A_1, A_2 \in \mathrm{End}(\mathbb{R}^n)$; this is why we need path-ordered exponentials in this problem.)

Therefore, F is indeed a functor.

An $\mathrm{End}(\mathbb{R}^n)$*-valued 1-form A is called a* **connection** *on the trivial vector bundle*

$$\pi: X \times \mathbb{R}^n \to X.$$

If $\psi(t)$ satisfies Equation (2), we say the vector $\psi(t)$ is **parallel transported** *along the path γ using the connection A. The linear operator $F(\gamma)$ is called the* **holonomy** *of the connection A along the path γ.*

All this stuff generalizes to the case of a connection on a nontrivial vector bundle

$$\pi \colon E \to X$$

except that now the functor F maps each point $x \in X$ to the **fiber of** E **over** x*, namely $E_x = \pi^{-1}(x)$. To handle this case, we choose an open cover of X such that E restricted to each open set is trivializable, and reduce the problem to the case treated above. After huffing and puffing, we get:*

Theorem 2. *Suppose A is a smooth connection on a smooth vector bundle $\pi \colon E \to X$ over a smooth manifold X. Then there is a unique functor*

$$F \colon \mathcal{P}(X) \to \mathrm{Vect}$$

such that:

- *For any object x of $\mathcal{P}(X)$, $F(x)$ is the fiber of E over x.*
- *For any morphism $\gamma \colon x \to y$ of $\mathcal{P}(X)$, $F(\gamma)$ is the holonomy of A along γ.*

The converse is not true: there are functors $F \colon \mathcal{P}(X) \to \mathrm{Vect}$ that don't come from connections on vector bundles! However, we can characterize the functors that do by means of three conditions:

- *$F(\gamma_1) = F(\gamma_2)$ when γ_2 is obtained by reparametrizing γ_1:*

  $$\gamma_2(t) = \gamma_1(f(t))$$

  for any monotone increasing function f.
- *$F(\gamma_2) = F(\gamma_1)^{-1}$ when γ_2 is a reversed version of γ_1:*

  $$\gamma_2(t) = \gamma_1(f(t))$$

  for any monotone decreasing function f.
- *$F(\gamma)$ depends smoothly on γ (in a certain precise sense).*

For some hints on how to prove this, try:

J. Barrett, Holonomy and path structures in general relativity and Yang–Mills theory, *Int. J. Theor. Phys.*, **30** (1991), 1171–1215.

If we drop the smoothness condition, we call F a **generalized connection**. *These play an important role in loop quantum gravity.*