Duals

Questions by: John C. Baez, November 4, 2004
Answers by: Toby Bartels[1], 2004 November 11

The concept of 'dual vector space' has a massive generalization in terms of category theory. It goes like this. . . .

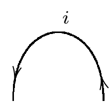
Suppose C is a monoidal category. An **adjunction** *in C is a quadruple (x, x^*, i, e) where:*

- *x and x^* are objects in C.*
- *$i\colon 1 \to x \otimes x^*$ and $e\colon x^* \otimes x \to 1$ are morphisms in C (called the* **unit** *and* **counit** *of the adjunction, respectively).*
- *The following diagrams commute:*

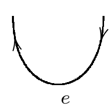
$$\begin{tikzcd}
1 \otimes x \arrow[r, "i\otimes 1"] \arrow[d, "\ell_x"'] & (x \otimes x^*) \otimes x \arrow[r, "a_{x,x^*,x}"] & x \otimes (x^* \otimes x) \arrow[d, "1\otimes e"] \\
x \arrow[rr, "r_x^{-1}"'] & & x \otimes 1
\end{tikzcd}$$

$$\begin{tikzcd}
x^* \otimes 1 \arrow[r, "1\otimes i"] \arrow[d, "r_{x^*}"'] & x^* \otimes (x \otimes x^*) \arrow[r, "a^{-1}_{x^*,x,x^*}"] & (x^* \otimes x) \otimes x^* \arrow[d, "e\otimes 1"] \\
x^* \arrow[rr, "\ell_{x^*}^{-1}"'] & & 1 \otimes x^*
\end{tikzcd}$$

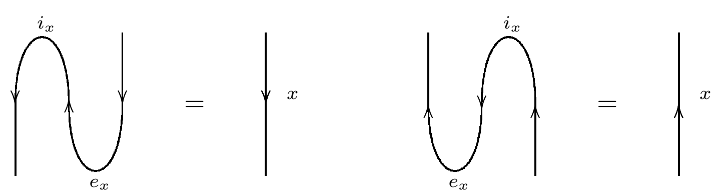
Aaron Lauda has dubbed the above commutative diagrams the **zig-zag identities**. *Why? The string diagram for the unit $i\colon 1 \to x \otimes x^*$ looks like this:*



where it is understood that the downward pointing arrow corresponds to x and the upward pointing arrow to x^*. Similarly, the counit $e\colon x^* \otimes x \to 1$ looks like this:



These string diagrams are reminiscent of the Feynman diagrams for the creation and annihilation of particle/antiparticle pairs! In this notation, the zig-zag identities simply say that we can straighten a zig-zag in a piece of string:



1. The category Vect_k has finite-dimensional vector spaces over a fixed field k as its objects and linear maps between these as its morphisms. Vect becomes a monoidal category with the usual tensor product of vector spaces and with the unit object $1 = k$.

Suppose $V \in \mathrm{Vect}_k$ and V^* is its dual, i.e. the space of all linear maps $f \colon V \to k$. Define $i_V \colon k \to V \otimes V^* \cong \mathrm{End}(V)$ by

$$i_V(\alpha) = \alpha\, 1_V$$

and define $e_V \colon V^* \otimes V \to k$ by

$$e_V(f \otimes v) = f(v).$$

a. Show that (V, V^*, i_V, e_V) is an adjunction.

I just have to check the commutative diagrams. To identify $\mathrm{End}(V)$ *with* $V \otimes V^*$*, let* $(e_i)_i$ *be a basis of* V*, with the dual basis* $(e^i)_i$ *of* V^**; then* 1_V *is identified with* $\sum_i (e_i \otimes e^i)$*, which in turn is abbreviated* $e_i \otimes e^i$*. Thus* $i_V(\alpha) = \alpha\, e_i \otimes e^i$*.*

Starting with $\alpha \otimes v \in 1 \otimes V$*, I go around to the right to get* $\alpha\,(e_i \otimes e^i) \otimes v$*, then* $\alpha\, e_i \otimes (e^i \otimes v)$*, and then* $\alpha\, e_i \otimes v^i = \alpha v^i e_i \otimes 1 = \alpha \otimes 1$*, where the* v^i *are the components of* v *relative to the basis* $(e_i)_i$*. Going around to the left, I get* αv *and then* $\alpha v \otimes 1$*. Thus, this diagram commutes.*

Then starting with $f \otimes \alpha \in V^* \otimes 1$*, I go around to the right to get* $\alpha\, f \otimes (e_i \otimes e^i)$*, then* $\alpha\,(f \otimes e_i) \otimes e^i$ *and then* $\alpha\, f_i \otimes e^i = \alpha\, 1 \otimes f_i e^i = \alpha \otimes f$*, where the* f^i *are the components of* f *relative to the basis* $(e^i)_i$*. Going around to the left, I get* $\alpha\, f$ *and then* $\alpha \otimes f$*. Thus, this diagram also commutes.*

Therefore, this is an adjunction.

b. What goes wrong when V is infinite-dimensional?

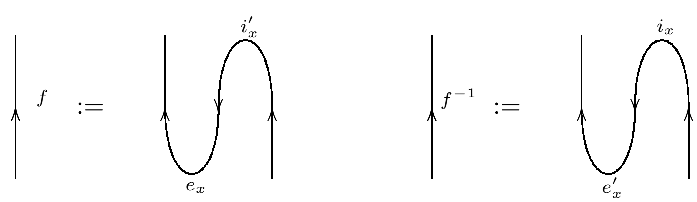
Now I can no longer identify $\mathrm{End}(V)$ *with* $V \otimes V^*$*, so I can't define* i_V*. More specifically, there is still a monomorphism from* $V \otimes V^*$ *to* $\mathrm{End}(V)$*, but its image consists only of the operators of finite rank. Since* 1_V *now has infinite rank, the construction fails.*

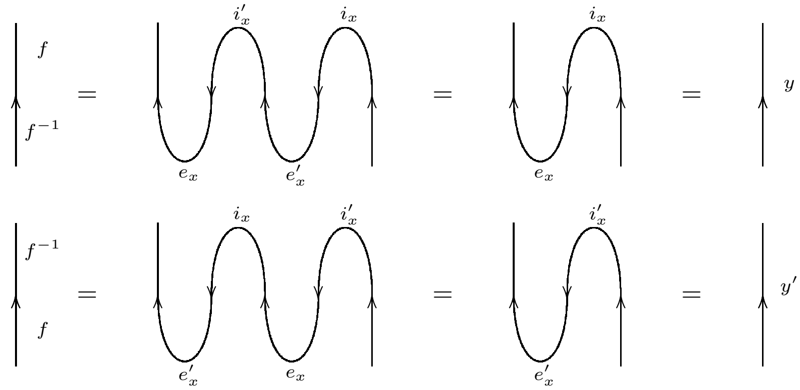
In the above situation we often call V^* *'the' dual of* V*, but one should be a bit careful. After all, the precise definition of 'linear map' depends on the definition of 'function', and different people use slightly different definitions of 'function' — for example, by saying a function is a set of ordered pairs, but using different definitions of 'ordered pair', such as Norbert Wiener's original 1914 definition* $(x, y) = \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$*, Kazimierz Kuratowski's more efficient 1921 definition* $(x, y) = \{\{x\}, \{x, y\}\}$*, or his brother Zreimizak's 1922 definition* $(x, y) = \{\{y\}, \{y, x\}\}$*. (Tragically, Kazimierz and Zreimizak killed each other in a foolish swordfight over this issue in 1923.)*

So, if we were being incredibly nitpicky, we might call V^* *'a' dual of* V*. The concept of adjunction makes this more precise, by saying exactly what a dual should be like — at least in the finite-dimensional case. And the really nice thing is that we can prove that any two duals of the same object are isomorphic in a god-given way:*

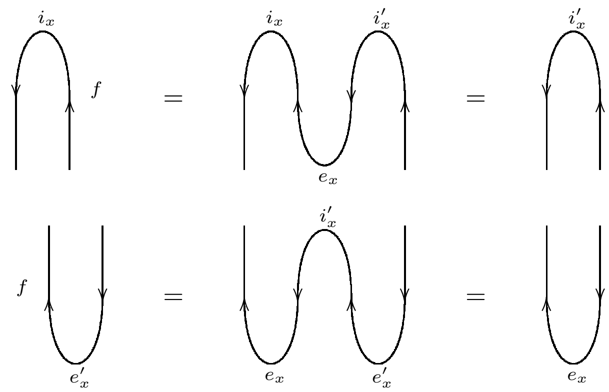
2. Suppose x is an object in the monoidal category C and (x, y, i, e) and (x, y', i', e') are adjunctions.

a. Construct an isomorphism $f \colon y \to y'$.





b. Describe the sense in which the isomorphism $f\colon y \to y'$ makes (x, y, i, e) and (x, y', i', e') into isomorphic adjunctions.



(Hint: it's easiest to do these using string diagrams.)

This result means we're allowed to speak of 'the dual' of x as long as we use the word 'the' in its official category-theoretic sense. In set theory, we're allowed to speak of **the** *element with some property whenever such an object exists and any two elements with this property are equal. In category theory, we're allowed to speak of* **the** *object equipped with some stuff whenever such an object exists and any two objects equipped with this stuff are isomorphic in a specified way.*

Finally, let's show that monoidal functors automatically preserve duals of objects:

3. Suppose C and D are monoidal categories and $F\colon C \to D$ is a monoidal functor. Show that if (x, y, i, e) is an adjunction in C, there is an adjunction in D making $F(y)$ into the dual of $F(x)$.

Since F is a monoidal functor, it comes equipped with isomorphisms $\Phi_{x,y}\colon F(x) \otimes_D F(y) \to F(x \otimes_C y)$, $\Phi_{y,x}\colon F(y) \otimes_D F(x) \to F(y \otimes_C x)$, and $\phi\colon 1_D \to F(1_C)$. Using these, let $i_D\colon 1 \to F(x) \otimes F(y)$ be the composition $\phi\, F(i)\, \Phi_{x,y}^{-1}$, and let $e_D\colon F(y) \otimes F(x) \to 1$ be $\Phi_{y,x}\, F(e)\, \phi^{-1}$.

Then this diagram commutes:

$$
\begin{tikzcd}
1 \otimes F(x) \arrow[r, "i_D \otimes 1"] \arrow[d, "\ell_{F(x)}"'] & (F(x) \otimes F(y)) \otimes F(x) \arrow[r, "a_{F(x),F(y),F(x)}"] & F(x) \otimes (F(y) \otimes F(x)) \arrow[d, "1 \otimes e_D"] \\
F(x) \arrow[rr, "r^{-1}_{F(x)}"'] & & F(x) \otimes 1
\end{tikzcd}
$$

because it may be broken down as:

$$\begin{tikzcd}
1_D \otimes F(x) \arrow[rr, "i_D \otimes F(1_x)"] \arrow[dr, "\phi \otimes F(1_x)"] \arrow[dddr, swap, "\ell_{F(x)}"] & & (F(x) \otimes F(y)) \otimes F(x) \arrow[d, "\Phi_{x,y} \otimes F(1_x)"] \arrow[drrr, "\alpha_{F(x),F(y),F(x)}"] & & & \\
& F(1_C) \otimes F(x) \arrow[r, "F(i) \otimes F(1_x)"] \arrow[d, swap, "\Phi_{1_C,x}"] & F(x \otimes y) \otimes F(x) \arrow[d, "\Phi_{x \otimes y, x}"] & & & F(x) \otimes (F(y) \otimes F(x)) \arrow[dl, swap, "F(1_x) \otimes \Phi_{y,x}"] \arrow[ddd, "F(1_x) \otimes e_D"] \\
& F(1_C \otimes x) \arrow[r, swap, "F(i \otimes 1_x)"] \arrow[d, "F(\ell_x)"] & F((x \otimes y) \otimes x) \arrow[r, "F(\alpha_{x,y,x})"] & F(x \otimes (y \otimes x)) \arrow[d, swap, "F(1_x \otimes e)"] & F(x) \otimes F(y \otimes x) \arrow[l, swap, "\Phi_{x,y \otimes x}"] \arrow[d, "F(1_x \otimes e)"] & \\
& F(x) & & F(x \otimes 1_C) \arrow[ll, "F(r_x)"] & F(x) \otimes F(1_C) \arrow[l, "\Phi_{x,1_C}"] & \\
& & & & & F(x) \otimes 1_D \arrow[ul, swap, "F(1_x) \otimes \phi"] \arrow[ullll, "r_{F(x)}"]
\end{tikzcd}$$

Similarly, this diagram commutes:

$$\begin{tikzcd}
F(y) \otimes 1 \arrow[r, "1 \otimes i_D"] \arrow[d, swap, "r_{F(y)}"] & F(y) \otimes (F(x) \otimes F(y)) \arrow[r, "a^{-1}_{F(y),F(x),F(y)}"] & (F(y) \otimes F(x)) \otimes F(y) \arrow[d, "e_D \otimes 1"] \\
F(y) \arrow[rr, swap, "\ell^{-1}_{F(y)}"] & & 1 \otimes F(y)
\end{tikzcd}$$

because it may be broken down into another huge diagram, which I'll skip.

[Note: I did not use string diagrams for this, because it seems to me that their validity depends on already *having an adjunction.]*

Therefore, $(F(x), F(y), i_D, e_D)$ *is an adjunction in the monoidal category* D.

(Hint: when F *is a strict monoidal functor this adjunction in* D *is just* $(F(x), F(y), F(i), F(e))$, *but in general we need to keep track of the fact that* F *preserves the tensor product and unit object only up to specified isomorphisms.)*