In our quest to build 2d TQFTs, we start with a vector space $A$ equipped with a binary operation

$$m: A \otimes A \rightarrow A$$

which we draw as

![Diagram](image)

We demand that $m$ be associative:

![Diagrams](images)

to get the 2-2 move.

To deal with triangles like this:

![Diagram](image)

we want an isomorphism $A \cong A^*$, to get

$$A \cong A^* \xrightarrow{m^+} A^* \otimes A^* \cong A \otimes A$$
For this, we demanded that a certain God-given pairing
\[ g : A \otimes A \rightarrow C \]
be nondegenerate, i.e. that
\[ # : A \rightarrow A^* \]
\[ a \mapsto g(a \otimes -) \]
be an isomorphism. What's this God-given \( g \)? It's

\[ \varepsilon_A : A^* \otimes A \rightarrow C \]
\[ f \otimes v \mapsto f(v) \]

In fact, to get a 2d TQFT all we need is this: a vector space \( A \) with bilinear associative product \( m \) s.t. \( g \) is nondegenerate. Such a thing turns out to be a semisimple algebra!

What's the meaning of \( g \)? We saw that given any linear operator \( T : V \rightarrow V \),
\[ \text{tr} (T) = \]
Note: left multiplication by $a \in A$ defines a linear operator:

\[ L_a : A \to A \]
\[ b \mapsto ab \]

which we draw as

\[ \begin{array}{c}
\vdots \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad a \\
\end{array} \begin{array}{c}
\vdots \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad b \\
\end{array} \]

where

\[ \begin{array}{c}
\vdots \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad \downarrow 1 \rightarrow a \\
\end{array} \]

is our notation for an element of $A$.

So:

\[ g(a \circ b) = \begin{array}{c}
\vdots \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad \circ \\
\end{array} \begin{array}{c}
\vdots \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad b \\
\end{array} = tr(L_b L_a) = tr(L_a L_b) \]

(This would be called the "Killing form" if $A$ were a Lie algebra & $m = [\cdot, \cdot]$ A Lie algebra is called semisimple if $g$ is nondegenerate.)
Since we're assuming \( \gamma \) is nondegenerate, we have one isomorphism
\[
\# : A \to A^* 
\]
which we'll use to identify \( A \& A^* \). So, we won't draw arrows on our string diagrams anymore! So:

\[
\begin{align*}
\text{m} : A \& A &\to A \\
\text{m}^+ : A^* &\to A^* \& A^* 
\end{align*}
\]
gives
\[
\text{m}^+ : A &\to A \& A
\]

Also, we'll define

\[
\begin{array}{c}
\begin{array}{c}
\text{v}
\end{array}
\end{array} 
\]

so that we can draw

\[
\begin{array}{c}
\begin{array}{c}
g =
\end{array}
\end{array}
\]

or just

\[
\begin{array}{c}
\begin{array}{c}
g = 
\end{array}
\end{array}
\]
or:

\[
\begin{array}{c}
\text{or:} \\
\Uparrow \quad g \\
\Uparrow \\
\end{array}
\]

Don't worry – this can be proved to be harmless!

So: we have a vector space A with

\[
\begin{array}{c}
\Uparrow \\
\end{array}
\]

such that:

\[
\begin{array}{c}
\Uparrow \\
\Uparrow \\
(associativity)
\end{array}
\]

&

\[
\begin{array}{c}
\Uparrow \\
\Uparrow \\
\end{array}
\]

as well as the usual

\[
\begin{array}{c}
\Uparrow \\
\Uparrow \\
\Uparrow \\
\end{array}
\]

\[
\Uparrow = 1 = W. \quad (\text{nondegeneracy})
\]
We saw that
\[
\begin{align*}
\Upsilon &= \Upsilon \\
\end{align*}
\]
becomes the 2-2 move when we draw its Poincaré dual:
\[
\begin{align*}
\square &= \square \\
\end{align*}
\]
But what about the 1-3 move?
\[
\begin{align*}
\triangle &= \triangle \\
\end{align*}
\]
i.e.
\[
\begin{align*}
\n &= \n \\
\end{align*}
\]
Claim: this follows from the rules we have!
Let's draw the Poincaré dual of
\[
\begin{align*}
\Upsilon &= \Upsilon \circ \Upsilon \\
\end{align*}
\]
i.e.
\[
\begin{align*}
\psi &= \psi \\
\end{align*}
\]
the bubble move, e.g.

```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```

**Theorem:** Given the 2-2 move, the 3-1 move is equivalent to the bubble move.

```
bubble \Rightarrow 3-1
```

```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```

or

```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```


```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```

```
3-1 \Rightarrow \text{bubble}
```

```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```

```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```

```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```

```
\begin{center}
\begin{array}{c}
\begin{array}{c}
\quad = \quad \\
\end{array}
\end{array}
\end{center}
```