Quantum Gravity Seminar
Homework 2

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Given a category C with finite products, show it can be turned into a monoidal category.

Universal property of products

Given objects $A_1, A_2, \ldots, A_n$ in C, for some n, there exists an object A, called the product, with morphisms $\pi_i : A \to A_i$, for each i, such that for every object X in C with morphisms $\phi_i : A \to A_i$, for each i, there exists a unique morphism $\phi : X \to A$ such that

commutes. i.e., $\pi_i \circ \phi = \phi_i$, for all i.

Isomorphism of products

It will be useful to first show that any two products of a given collection of objects are isomorphic.
Consider two products D and D' of the objects A and B, each with projection maps to A and B.

By the universal property of products there exist unique maps $\phi : D \to D'$ and $\phi' : D' \to D$ such that each triangle commutes. There also exists a unique map

from D to D which makes the square commute. It is easily checked that the composition $\phi' \circ \phi$ makes the square commute. Since the identity satisfies this property we have $\phi' \circ \phi = 1_D$. Thus, D and D' are isomorphic. This proof generalizes easily to products of any finite number of objects.

The product of no objects is terminal

Consider the product of no objects, which we call 1. We would like to show that this is a terminal object in our category, i.e., for every object X in $\boldsymbol{C}$ there is exactly one morphism to 1.

```
X
|
| !
∀
1
```

For any object the diagram above vacuously satisfies the hypotheses of the universal property of products, so there exists a unique map into 1.
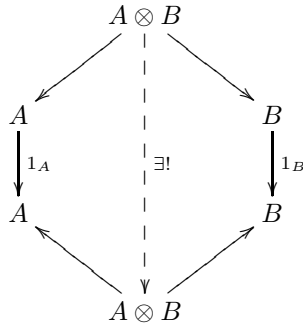
The tensor functor

We define the map $\otimes\colon \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$ both on objects and morphisms. Given a pair of objects A and B and assuming the axiom of choice, the tensor map chooses a product of A and B with projection maps to A and B. (In what follows, the arrows which are these projections will be not be labelled since it is clear what they are.) Since we have already shown that all products of A and B are isomorphic, we will call the chosen product $A \otimes B$ and the map of objects is well-defined up to isomorphism. Given a pair of morphisms $f\colon A \to A'$ and $g\colon B \to B'$, consider the diagram

and we define $\otimes(f, g)$ to be the unique map $f \otimes g\colon A \otimes B \to A' \otimes B'$ in the diagram.

The identity
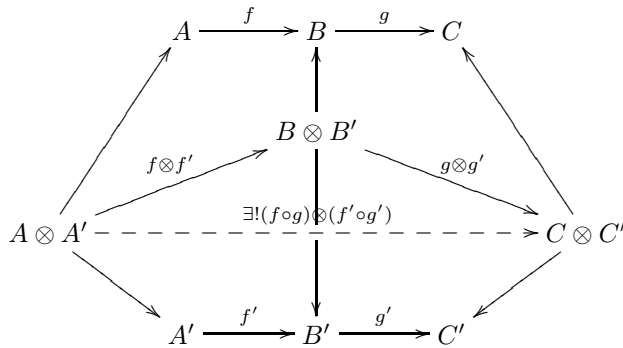
Similarly, if we consider the identity maps $1_A\colon A \to A$ and $1_B\colon B \to B$, the following diagram gives the unique morphism $1_A \otimes 1_B$ from A⊗B to A⊗B such that the diagram commutes.

$$
\begin{array}{c}
A \otimes B \\
\swarrow \quad \vdots \quad \searrow \\
A \qquad\qquad B \\
\downarrow{\scriptstyle 1_A} \quad \vdots {\scriptstyle \exists!} \quad \downarrow{\scriptstyle 1_B} \\
A \qquad\qquad B \\
\nwarrow \quad \vdots \quad \nearrow \\
A \otimes B
\end{array}
$$

Since the identity map $1_{A\otimes B}$ satisfies this diagram, we have $1_A \otimes 1_B = 1_{A\otimes B}$.

Composition

Given morphisms $f\colon A \to B$, $g\colon B \to C$, $f'\colon A' \to B'$, $g'\colon B' \to C'$. We want to show $\otimes$ respects composition of morphisms.

$$
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g} C \\
B \otimes B' \\
{\scriptstyle f\otimes f'} \qquad {\scriptstyle g\otimes g'} \\
A \otimes A' \dashrightarrow^{\exists!(f\circ g)\otimes(f'\circ g')} C \otimes C' \\
A' \xrightarrow{f'} B' \xrightarrow{g'} C'
\end{array}
$$

The definition of tensors of morphisms given above tells us that each small square of this diagram commutes giving $f \otimes f'$ and $g \otimes g'$. The full diagram itself defines $(g \circ f) \otimes (g' \circ f')$ by the same tensor definition. Since all the small squares commute, $g \otimes g' \circ f \otimes f'$ satsifies the commutivity of the larger square. Thus, by uniqueness, we must have $(g \circ f) \otimes (g' \circ f') = (g \otimes g') \circ (f \otimes f')$.

The unitors

Since $1 \otimes A$ and A are both products of 1 and A, we have an isomorphism $l_A\colon 1 \otimes A \to A$ for every object A in our category given by the diagram

$$
\begin{array}{c}
1 \\
\nearrow^{!} \quad \uparrow^{!_A} \\
1 \otimes A \xrightarrow{\;\;\exists !\, l_A\;\;} A \\
\searrow \qquad \downarrow^{1_A} \\
A
\end{array}
$$

Similarly, we have an isomorphism $r_A \colon A \otimes 1 \to A$. Note that the above diagram says that l_A must be the very same projection chosen by the functor from $A \otimes 1$ to A. The same is true for r_A.

Naturality

Given the functors $F \colon \boldsymbol{C} \to \boldsymbol{C}$ defined by $A \longmapsto 1 \otimes A$, $f \longmapsto 1 \otimes f$ and the identity functor, we now show that l_A is natural. That is, given objects A and B in C and a morphism f in C from A to B, we want the following diagram to commute:

$$
\begin{array}{ccc}
F(A) = 1 \otimes A & \xrightarrow{\;F(f) = 1 \otimes f\;} & 1 \otimes B = F(B) \\
\downarrow^{l_A} & & \downarrow^{l_B} \\
Id(A) = A & \xrightarrow{\;Id(f) = f\;} & B = Id(B)
\end{array}
$$

But extending this diagram slightly we have the diagram which defined the tensor functor on morphisms

$$
\begin{array}{ccc}
1 & \xrightarrow{\;\;!\;\;} & 1 \\
\uparrow & & \uparrow \\
1 \otimes A & \xrightarrow{\;1 \otimes f\;} & 1 \otimes B \\
\downarrow^{l_A} & & \downarrow^{l_B} \\
A & \xrightarrow{\;\;f\;\;} & B
\end{array}
$$

which commutes. The bottom square is the same diagram as our naturality square above and the squares commute as desired. A similar argument gives us that r_A is natural.

The associators

We now define ismorphisms of products called the associators. Given objects A, B and C in $\boldsymbol{C}$, we consider the product diagram

$$(A \otimes B) \otimes C$$

$$A \otimes B \qquad\qquad C$$

$$A \qquad B \xleftarrow{\;\exists!\alpha_{A,B,C}\;} B \otimes C$$

$$A \otimes (B \otimes C)$$
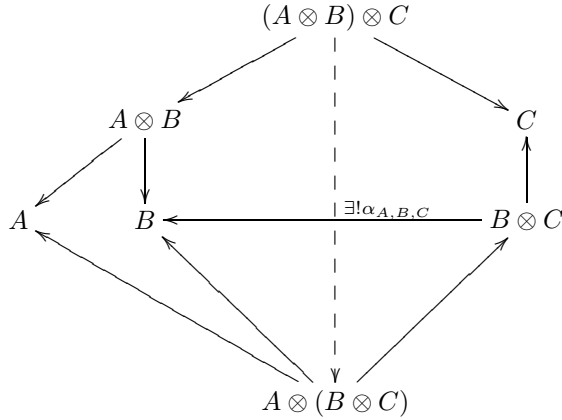
and call the unique map $\alpha_{A,B\otimes C}$ the associator. As with the other morphisms given from product diagrams, this map is an isomorphism.

Naturality

As with the unitors, we need to show the associator is a natural isomorphism. So given functors F and G which take objects A, B, and C to $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ respectively, and morphisms $f\colon A \to A'$, $g\colon B \to B'$, and $h\colon C \to C'$ to $(f \otimes g) \otimes h$ and $f \otimes (g \otimes h)$ respectively, we must show the following diagram commutes.

$$
\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{(f\otimes g)\otimes h} & (A' \otimes B') \otimes C' \\
\downarrow{\scriptstyle \alpha_{A,B,C}} & & \downarrow{\scriptstyle \alpha_{A',B',C'}} \\
A \otimes (B \otimes C) & \xrightarrow{f\otimes(g\otimes h)} & A' \otimes (B' \otimes C')
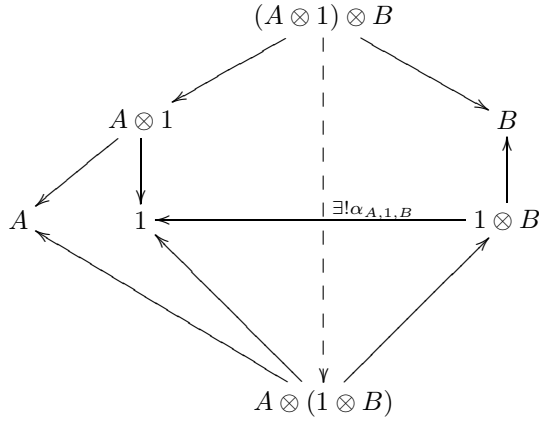\end{array}
$$

Each of the morphisms in this square are defined as the unique morphism that makes some diagram commute. These diagrams use the universal property of products with projections onto either A, B, and C or A', B', and C'. Since we have morphisms between each object and its primed partner, respectively, it can be shown that this diagram is made up of several smaller commuting diagrams, therefore it commutes. The explicit calculations are left out since they are cumbersome and not very instructive.
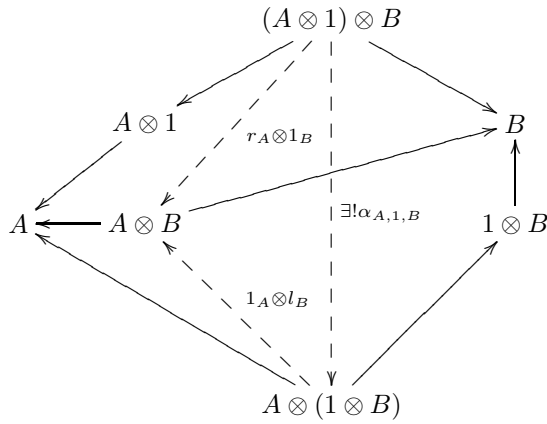
The unit laws

We want the following diagram to hold.

$$
\begin{array}{ccc}
(A \otimes 1) \otimes B & \xrightarrow{\;\alpha_{A\otimes 1,B}\;} & A \otimes (1 \otimes B) \\
 & \searrow{\scriptstyle r_A\otimes 1_B} \quad \swarrow{\scriptstyle 1_A\otimes l_B} & \\
 & A \otimes B &
\end{array}
$$

The appropriate definition for $\alpha_{A,1,B}$ comes from the following diagram.

$$(A \otimes 1) \otimes B$$

$$A \otimes 1 \qquad B$$

$$A \qquad 1 \xleftarrow{\ \exists! \alpha_{A,1,B}\ } 1 \otimes B$$

$$A \otimes (1 \otimes B)$$

So, we have

$$(A \otimes 1) \otimes B$$

$$A \otimes 1 \qquad B$$

$$\begin{array}{c} r_A \otimes 1_B \end{array}$$

$$A \xleftarrow{\ } A \otimes B \qquad \exists! \alpha_{A,1,B} \qquad 1 \otimes B$$

$$1_A \otimes l_B$$

$$A \otimes (1 \otimes B)$$

$r_A \otimes 1_B$ and $1_A \otimes l_B$, respectively, make the top and bottom of the diagram commute since these are the diagrams in the respective product diagram definitions. Since the associator also makes this diagram commute, we have that all smaller diagrams here commute. $(1_A \otimes l_B) \circ \alpha_{A,1,B}$ and $r_A \otimes 1_B$ are both maps from $(A \otimes 1) \otimes B$ to $A \otimes B$ and they both make the following portion of the above diagram commute.

$$(A \otimes 1) \otimes B$$

$$A \otimes 1 \qquad B$$

$$A \xleftarrow{\ } A \otimes B$$
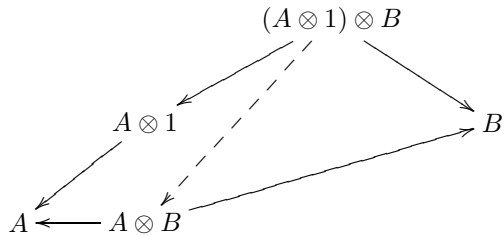
So, the two morphisms are equal and the desired diagram commutes.

The pentagon equation

Finally, given objects A, B, C, and D, the associators must satisfy the following diagram.

$$((A \otimes B) \otimes C) \otimes D$$

$$\alpha_{A \otimes B, C, D} \qquad \qquad \alpha_{A,B,C} \otimes 1_D$$

$$(A \otimes B) \otimes (C \otimes D) \qquad \qquad (A \otimes (B \otimes C)) \otimes D$$

$$\alpha_{A,B,C \otimes D} \qquad \qquad \alpha_{A,B \otimes C,D}$$

$$A \otimes (B \otimes (C \otimes D)) \xleftarrow{\quad 1_A \otimes \alpha_{B,C,D} \quad} A \otimes ((B \otimes C) \otimes D)$$

Each of the objects in the pentagon is a product of the four objects $A, B, C,$ and D with the appropriate projection maps. All of the morphisms in the pentagon equation are associators or tensor products of associators and identity maps. Therefore, they all satisfy commutative product diagrams with projections onto each of the four objects. Drawing each of these product diagrams gives a tangle of morphisms and triangles which make up the pentagon. Since each diagram comprising the pentagon commutes, it is a simple (yet messy) exercise to see that the pentagon actually commutes.