A Spring in Imaginary Time

Math 241 Homework
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One of the stranger aspects of Lagrangian *dynamics* is how it turns into *statics* when we replace the time coordinate t by it — or in the jargon of physicists, when we 'Wick rotate' to 'imaginary time'! People usually take advantage of this to do interesting things in the context of *quantum mechanics*, but the basic ideas are already visible in *classical mechanics*. So, let's look at them!

1. Suppose you have a spring in $\mathbb{R}^n$ whose ends are held fixed, tracing out a curve

$$q\colon [s_0, s_1] \to \mathbb{R}^n$$

with endpoints

$$q(s_0) = a, \qquad q(s_1) = b.$$

Suppose the spring is put into a potential

$$V\colon \mathbb{R}^n \to \mathbb{R}$$

(perhaps due to gravity, but not necessarily). What curve will the spring trace out when it is in equilibrium?

Hint: Hooke's law says that a stretched spring has energy proportional to the square of how much it is stretched. Here this is true of each little piece of the spring, so its total energy due to stretching will be

$$\frac{k}{2}\int_{s_0}^{s_1} \dot{q}(s)\cdot\dot{q}(s)\,ds$$

for some 'spring constant' k. But in addition, each little piece will acquire energy due to the potential V at that point, so the spring will also have potential energy

$$\int_{s_0}^{s_1} V(q(s))\,ds.$$

The total energy of the spring is thus:

$$E = \int_{s_0}^{s_1} \left(\frac{k}{2}\dot{q}(s)\cdot\dot{q}(s) + V(q(s))\right)\,ds.$$

Our study of statics has taught us that in equilibrium, a static system minimizes its energy, or at least finds a critical point. So, set

$$\delta E = 0$$

for all allowed variations δq of the path, and work out the differential equation this implies for q.

Answer: A variation $\delta q(s)$ produces the variation

$$\delta E = \int_{s_0}^{s_1} \left(k\dot{q}(s)\cdot\delta\dot{q}(s) + \nabla V(q(s))\cdot\delta q(s)\right)\,ds$$

in E. The first term in the integrand can be integrated by parts, and, as $\delta q(s_0) = \delta q(s_1) = 0$, this yields

$$\delta E = \int_{s_0}^{s_1} \left(-k\ddot{q}(s)\cdot\delta q(s) + \nabla V(q(s))\cdot\delta q(s)\right)\,ds = \int_{s_0}^{s_1} \left(-k\ddot{q}(s) + \nabla V(q(s))\right)\cdot\delta q(s)\,ds.$$

If δE is to vanish for all variations $\delta q(s)$, we must have

$$-k\ddot{q}(s) + \nabla V(q(s)) = 0,$$

which is the required differential equation for $q(s)$.

2. Suppose the spring is in a constant downwards gravitational field in $\mathbb{R}^3$, so that

$$V(x, y, z) = mgz$$

where m is the mass density of the spring and g is the acceleration of gravity (9.8 meters/second^2). What sort of curve does the spring trace out, in equilibrium?

Answer: As $\nabla V(x, y, z) = mg\hat{z}$, the equations for x, y, and z are

$$\ddot{x}(s) = 0, \qquad \ddot{y}(s) = 0, \qquad \text{and} \qquad \ddot{z}(s) = \frac{mg}{k}.$$

The solutions are

$$\begin{aligned} x(s) &= u_x s + v_x \\ y(s) &= u_y s + v_y \\ z(s) &= u_z s + v_z + \frac{mg}{2k} s^2, \end{aligned}$$

where u_i and v_i denote constants easily calculated from s_0, s_1, a, and b. These solutions parametrically define a parabola in $\mathbb{R}^3$.

3. The calculation in problem 1 should remind you strongly of the derivation of the Euler-Lagrange equations for a particle in a potential. To heighten this analogy, take the energy

$$E = \int_{s_0}^{s_1} \left(\frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right) ds.$$

and formally replace the parameter s by it, replacing the real interval $[s_0, s_1] \subset \mathbb{R}$ by the imaginary interval $[t_0, t_1] \subset i\mathbb{R}$, where $it_i = s_i$. Show that up to some constant factor, the *energy* of the static spring becomes the *action* for a particle moving in a potential.

Answer: Formally replacing s by it in our expression for E yields

$$E = \int_{it_0}^{it_1} \left(\frac{k}{2} \dot{q}(it) \cdot \dot{q}(it) + V(q(it)) \right) d(it).$$

Using

$$\frac{d}{dt} q(it) = i\dot{q}(it),$$

we can write E in the form

$$E = \int_{it_0}^{it_1} \left(-\frac{k}{2} \frac{d}{dt} q(it) \cdot \frac{d}{dt} q(it) + V(q(it)) \right) d(it) = -i \int_{it_0}^{it_1} \left(\frac{k}{2} \frac{d}{dt} q(it) \cdot \frac{d}{dt} q(it) - V(q(it)) \right) d(t).$$

This quantity is proportional to the action of a particle of mass k moving in a potential $V(q(it))$.

4. Fill in the blanks in this analogy:

STATICS	DYNAMICS
Principle of Least Energy	Principle of Least Action
spring	particle
energy	$-i\times$action
elastic energy density	kinetic energy
potential energy density	potential energy
spring constant k	mass m

5. What particular dynamics problem is the statics problem in 2 analogous to? How is the solution to the statics problem related to the solution of this dynamics problem?

Answer: The statics problem in 2 is analogous to the dynamical problem of a mass moving in a uniform gravitational field subject to conditions on its position at fixed times t_0 and t_1. The solution to the dynamics problem, also a parabolic curve, can be recovered from that of the statics problem by formally replacing s by it.

Note that while in the statics problem the parabola 'hangs down,' in the dynamics problem the parabola 'arches up'!

6. What does Newton's law $F = ma$ become if we formally replace t by $s = it$?

Hint: by 'formally'*, I'm suggesting that you shouldn't think too much about what this actually means! It's a good thing to think about, but don't let that stop you from solving what's meant to be a quick and easy problem.*

Answer: This formal replacement only affects the second derivative inherent in a, and the result is $F = -m\frac{d^2}{ds^2}q$. One way to view this (at least in the one dimensional case $q\colon [s_0, s_1] \to \mathbb{R}$) is as an analogue of the stretched elastic string free to oscillate longitudinally, wherein a restoring force is proportional to the second derivative of displacement with respect to length along the string. In this case m plays the role of an elastic constant.

In short, working in imaginary time replaces $F = ma$ by $F = -ma$.