John Baez  Matrix Mechanics

Say $X, Y, Z$ are (finite) sets.

Heisenberg was interested in physical processes that take one physical system (be a set of states) to another. Then maybe one doesn't know definitely which state goes to which one. So we write down an amplitude for each possibility, e.g.

\[ \begin{array}{c}
   x' \\
   \downarrow \\
   x \\
   \downarrow \\
   x''
\end{array} \]

For $x', y', x, y, x''$ etc.

Then $F_{x' x} F_{y' y} F_{x x''} F_{y y''} \in \mathbb{C}$

b) What about composing two processes?

\[ \begin{array}{c}
   x \\
   \downarrow \\
   y \\
   \downarrow \\
   z
\end{array} \]

Then $(H_{12} \circ (G \circ F))_{x z} = \sum_{y \in Y} G_{y z} F_{x y} (\text{all entries are } \mathbb{C})$

so this really is (the transpose of) matrix multiplication.
Finally, we deal with Hilbert space.

(C) This means that we’re doing \( F : X \times Y \to C \) etc.

Moreover, we really are only using \( (C, +, \cdot) \), not \(-, \cdot\).

So, we’re not using the field axioms, just the ring axioms.

Let’s do matrix mechanics with other rigs—e.g., the Boolean rig.

Then \( F : X \times Y \) is given by its possibility matrix.

Each edge stands for a "+".

\[ + \rightarrow \text{OR} \quad \_ \rightarrow \text{AND} \]

and now the ("graph of") \( F \) is just that if a relation

\[ F : X \times Y \to \{0, \beta\} \]

a "linear" \( F : \{0, \beta\}^X \to \{0, \beta\}^Y \).

and \( \{0, \beta\}^X \) are also called subsets of \( X \).

(3) One can bring back \( \beta \)-and Hecke operators!—from \( \beta \).

Given a relation \( F : X \times Y \to \{0, \beta\} \), you can interpret it as a linear operator

\[ F : X \times Y \to C \]

using \( \nu : \{0, \beta\} \to C \).

If \( G \) acts on \( X \) \& \( Y \), \( F \) is \( G \)-equivariant

then \( G \) acts on the form \( \text{def} \left( X \times C \right)_G \) and
The Hecke operators

\[
\text{then } F : X \times Y \to C = \tilde{F} : C \times C \to C \text{ is also } G\text{-equivariant.}
\]

\(\square\) The only problem in \(F \to \tilde{F}\) is \(\tilde{F}_1 \tilde{F}_2 = \tilde{F}_1 \tilde{F}_2\)?

Because we do know that \(1 = 1 \quad (1 : X \times X \to C = \delta_{XX})\)

So if \(\tilde{F}_1 \tilde{F}_2 = \tilde{F}_1 \tilde{F}_2\), then we could say that this is

\[
\Rightarrow \quad \sim : (\text{relations between}) \to (\text{operators on fin. dim. vector spaces})
\]

\(\text{as a functor.} \quad \Box\)

\(\square\)

So, is it? NO because \(v : \{0,1\} \to C \text{ is NOT a}

\[
\text{group homomorphism.}
\]

It preserves AND \& but NOT OR \& \(1 \lor 1 \neq 1 \quad \frown\)

\(\square\)

Example

\[
\begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}
\]

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}
\]

\(\text{FIN-REL} \quad \Box\)

\(\text{FIN-VECT} \quad \Box\)

\(\square\)

So how does one correct this? Instead of \(\{0,1\}\text{-valued}

\[
\text{matrices, we could use } M_2\text{-valued matrices, since}
\]

\(\square\)
\[
\begin{align*}
\mathbb{N} \rightarrow C \text{ as a rig homomorphism} \\
\text{This is related to } \otimes, \boxplus \text{-valued matrices, via the rig homomorphism } \\
\mathbb{N} \rightarrow \mathbb{Q}, \mathbb{Z} \\
(\eta \rightarrow 1) \\
(0 \rightarrow 0)
\end{align*}
\]

We could also categorify this—then we use rig-categories instead of rigs! E.g., FIN-SET-valued matrices, also called spans!

\[
\begin{align*}
+ & \rightarrow \text{coproduct = disjoint union} \\
\times & \rightarrow \text{product = Cartesian product}
\end{align*}
\]

(1) **Note**: FIN-SET is not a rig, because it's not a set!

But modulo natural transformations (which quotient out FINSET so that we wind up into a set) No

\[
\text{FIN SET} \rightarrow \text{No} \rightarrow C \text{ is a 'rig map'}
\]

So, the GoF-picture from (6) above, would now change into

\[
(G \circ F)(v, k) = \frac{1}{j} C_{ijk} \times F_{ij}
\]

And then

\[
\hat{G} \circ \hat{F} = \hat{G} \circ \hat{F}
\]
A matrix \( F : X \times Y \to \text{FinSet} \) is also called a span, since we define
\[
S = \bigsqcup_{(i,j) \in X \times Y} F_{i,j}
\]
Define:
\[
S \xrightarrow{p_1} X \quad \text{and} \quad S \xrightarrow{p_2} Y.
\]
\[
\forall s \in S, \exists! (i,j) : s \in F_{i,j} \quad \text{Define} \quad p_1(s) = i, \quad p_2(s) = j.
\]

In other words, \( S[F_{i,j}] = \{ \text{arrows } (X \to Y) \} \), and
\[
\begin{align*}
F_{i,j} &= \{ s \in S : p_1(s) = i, \quad p_2(s) = j \}.
\end{align*}
\]

In this notation, does matrix multiplication look like?

In fact, \( ST = \emptyset \) if \( S \) and \( T \) are disjoint, \( S \cap T \neq \emptyset \). So this is the pullback.
This makes spans seem like morphisms in some category whose objects are finite sets, since we can now compose.

So let us think of all the data that we do have—

and what we need to make it a category.

- $\text{FIN-SET}^\text{op} = \text{finite sets for objects}$

- Span of finite sets: $S : X \rightarrow Y$, i.e. $S$

- Composition via pullbacks

- Identity "morphisms"

What about associativity? NOT equal, but isomorphic!

$(ST)U \cong S(TU)$

This is essentially because pullbacks are not equal but $\cong$. In general, if we're forming something in two different ways, and these ways use some universal objects, we should expect both outputs not necessarily to be equal, but isomorphic!
So these really is a bi-category here if:

- finite sets as objects
- spans of finite sets
- isomorphism of spans

Equivalently, there's a category FIN-SPAN, if:

- fin sets as objects
- isomorphism classes of spans as morphisms

And we're now getting a functor \( \text{FIN-SPAN} \to \text{FIN-Vet} \)

So this is a good thing, given our initial goal!

C. T-structures of spans: Morphisms of spans should be

\[
S \to T
\]

\[
\begin{array}{c}
\times \\
\times
\end{array}
\]

\[
X \times Y
\]

So we'll see next time onwards that this approach (via spans) to derived categories, Hecke operators, is the (a more) "correct" way to go.