Theorem (k any field) Let G be a finite group, and \(XY\) be finite \(G\)-sets. Let \(\text{hom}(XY)\) be the set of \(G\)-equivariant maps of spans of \(G\)-sets \(S\).

\[
S \xrightarrow{x \rightarrow y} (S, F, y, \text{and } G \text{ is compatible})
\]

This is a finitely generated \(\mathbb{N}_G\)-module, and we have a map

\[
\text{hom}(X, Y) \rightarrow \text{Hom}_G(k^X k^Y) = \sum \text{ intertwining operators } f : k^X \rightarrow k^Y
\]

for any field \(k\). The resulting map

\[
\text{hom}(X, Y) \rightarrow \text{Hom}_G(k^X k^Y)
\]

is onto, but not necessarily 1-1.

To state a better theorem, the "correct" decategorification needs to be used — not to come to the 1-category of classes of \(G\)-spaces, but to use the spans of groupoids, which we'll see later in this course.

We'll continue from where we left off last time — think of spans and the 9-Pascal triangle.

Recall from last time:
Here $X \& Y$ are spans of finite sets.

$$Y = \left\{ \begin{array}{c}
\binom{n}{k} F \\
\binom{n+1}{k} F
\end{array} \right\} \quad (\text{not proper set, not a genuine function, hence a span})$$

It's awkward to treat it as a span of $G$-sets since $GL(n, F)$ acts on $F^n$ but $GL(n+1, F)$ acts on $F^{n+1}$

(though this is just as awkward as $F^n \rightarrow F^{n+1}$)

We have a way, sort of around this via the embedding,

$$\phi : GL(n, F) \hookrightarrow GL(n+1, F)$$

But a better way is to work with groupoids.

2. (a) A groupoid is a category, where all morphisms are invertible. (Any category $C$ has an underlying groupoid $C_0$, with the same objects, but only isomorphisms as morphisms.)

Thus, the skeleton of any groupoid is a disjoint union, or coproduct, of groupoids with one element/object.

Defn A group is a 1-object groupoid.

E.g. $(\text{Fin Set})_0 \cong \coprod_{n \geq 0} n!$, $(\text{Fin Vect}_F)_0 \cong \coprod_{n \geq 0} GL(n, F)$
5. We can thus switch from thinking about spans of G-sets to spans of groupoids as follows:

Roughly speaking, a G-set $S$ can be turned into a groupoid via the weak quotient $S//G$,

and such that given $\text{G-set } S$, $\text{G'}-\text{set } S'$, $\varphi = \text{gph dom } G \rightarrow G'$,

and $\varphi : S \rightarrow S'$ s.t. $G \times S \xrightarrow{\varphi \times 1} G' \times S'$

and $\varphi : S \rightarrow S'$ s.t. $G \times S \xrightarrow{\varphi \times 0} G' \times S'$

and

all this should yield a functor $\Phi : S//G \rightarrow S'//G'$.

6. Let us first describe how to carry this out. Given a G-set $S$, how do we build a groupoid?

eg. $\mathbb{Z}/\mathbb{Z} = G$ acts on $S = \{e, g\}$

and one now has $\mathfrak{G} = S'$.

Say, this is the idea in general:

$S//G$ has
- **objects:** elements of $S$
- **morphisms:** $\left\{ s \xrightarrow{f} g(s) \right\} = G \times S$
- **composition:** $(h, g(s)) \cdot (g, t) = (hg, ts)$
- **inverse:** $(g, s)^{-1} = (g^{-1}, gs)$
Let us work out a specific example — for the case that we are interested in

\[ \text{GL}(n, F) \text{ action on } \binom{n}{k} \]  

So we get a groupoid \( \binom{n}{k} \), and objects are \( V = k\)-dim \( \leq F^n \)

Morphisms \( g : V \to V' \) are

\[ g \circ \text{GL}_k : g \cdot V \to V' \]

In fact, this groupoid is equivalent to \( \text{Flag}_{n,k} \)

= groupoid whose objects are \( n\)-dim vector spaces (over \( F \))

equipped with a \( k\)-dim subspace

and morphism \( f : (V \leq W) \to (V' \leq W') \) is

\[ V \xrightarrow{f} W \quad V' \xrightarrow{f} W' \]

re morphism \( f : W \to W' \)

so that \( f_{/V} = V' \)

This is the groupoid of "\( n\)-dim vector spaces equipped with a \( k\)-dim subspace".

More generally, given \( D = \left[ n_1 + n_2 + \cdots + n_k = n \right] = \text{flag} \)

\[ D(F^n) = \binom{n}{n_1, \ldots, n_k} = D\text{-flag} \text{ on } F^n \]

Now \( D(F^n) / \text{GL}(n, F) \leq \text{the groupoid} \) of \( F\text{-equipped } D\text{-flags} \)
Let us give this the name \( D(Vect_F) \)

Now, we can apply \( D \) to various things:

\[
\begin{align*}
\left( \eta_k \right) \in \mathbb{N}_0 & \quad \mapsto \quad \left( \eta_k \right) \in \mathbb{N}_0[F_p] \\
\downarrow & \\
\left( \eta_k \right) \in \text{FinSet} & \quad \mapsto \quad \left( \eta_k \right) \in \text{FinSet} \\
D(\eta_k) & \\
\downarrow & \\
\left( \eta_k \right) \in Gpd & \quad \mapsto \quad \left( \eta_k \right) \in Gpd \\
D(\text{Set}) & \\
\downarrow & \\
D(Vect_{F_p}) &
\end{align*}
\]

Next time, how \( S' \to S, G \to G', \text{Set} \to \text{Gpd} \) gives a functor \( \Phi : S/[G] \to S'/[G'] \), and thus a span of \( S \)-sets gives a span of groupoids.

The ultimate aim is to replace math/linear algebra with spans and groupoids -- the program called groupoidification.