1. **Theorem**

The \( \mathbb{VECT} \)-enriched category

\((\text{Finite } G\text{-sets, } \mathbb{H} \text{c}k \text{c}, \text{ Operate between } \text{these } \text{Free } G\text{'s})\)

is the decategorification of the \( \mathbb{CAT} \)-enriched category

\((\text{Finite } G\text{-sets, Category } \mathbb{H} \text{c}k \text{s})\)

2. John B. stated the theorem using the wrong kind of decategorification. He asked over \( \text{No; we'll work over } \mathbb{C} \).

3. We first explain what enrichment is, and how each version of "decategorification" relates to it.

If \( C \) is a category and \( D \) is a category with some notion \( \otimes \) in it, then \( \mathbb{D} \)-enriched category \( C\mathbb{D} \) is a category whose objects are objects of \( C \) (all of them) and whose hom-sets are now objects in \( D \).

The reason \( D \) should have some \( \otimes \) is so that \( \otimes \) compose morphisms in \( C\mathbb{D} \). Moreover, composition in \( D \)

\( \otimes \) is a morphism in \( D \).

\( \text{eg. } D = \text{VECT} \Rightarrow \text{ usual } \otimes = \otimes \text{ in } \mathbb{C} \),

\( D = \text{Cat} \Rightarrow \text{ direct product of Categories } \).

4. How is decategorification related to enrichment?
Given a \( \otimes \) -functor between \( \otimes \) categories \( A, B \),

we can define \( \mathbf{P}(\text{DECAT}) : A\text{-enriched CAT} \to B\text{-enriched CAT} \)

\( (C_B = \text{same objects as } C_A \text{ but } \text{Hom}_B = \text{DECAT}(\text{Hom}_A)) \)

(Of course, we also only work with "more complex" \( \to \) "less complex"
but really, this construction can be done more generally.)

(\textbf{Example}) Say \( 1 \) = Category with 1 object \( \& 1\) morphism.

Then one has \( \text{SET} \xrightarrow{\cdot} 1 \)

\( \text{all sets} \to (\cdot) \)

\( \text{all maps} \to (\text{id}_*) \)

What is \( \mathbf{C} = \text{SET} \xrightarrow{\cdot(1)} \text{C}_1 \)?

\( \text{ANS: } \mathbf{C}_1 \text{ has same \# of objects as } \text{SET} = \mathbf{C} \)

\( \text{Set-enriched Cat's are just usual categories!} \)

But all objects have only \( 1 \) map between them:

(\textbf{Lem:}) All objects are isomorphic.

(\textbf{Pro:}) \( f_{xy} : X \to Y \xrightarrow{\mu_X} X \) must be \( \text{id}_X \).
Conclusions: (Set of small 1-embedded categories) → (Set).

(Equivalence classes of (small) 1-embedded categories) ↔ \( \sum \emptyset \beta = TV = \text{Truth Values} \)

A \( TV \)-enriched category means every morphism set for the choice to be empty or nonempty, and every nonempty set just has one morphism in it.

Moreover, the category \( C_{TV} \) is itself a category. 

\[ |\text{Hom}(X,Y)| = 1 \quad \forall X, Y \]

Thus, \( C_{TV} \) is just a partial order on \( \text{Ob} \) \( C \).

We want \( \text{CAT} \rightarrow \text{C-Vect} \).

Actually, we do use \( D \): "nice CAT" → \( \text{C-VECT} \) to get \( P(D) \). "Nice" means such that it's actually a groupoid in disguise.

That is, it's the category of \( G \)-sets for some groupoid \( G \).
b) Groupoid theory is not very popular because people just break up the groupoid into its slices, and then just study its skeleton, so this comes back to group theory!

Also unpopular is the third homology of groups; as top spaces, groups have just 1 component, and
\[ H_3(\text{top space } X) = \mathbb{Z}^{\#\text{components}} \]

But now one can do $H_0(\text{groupoid}) = \mathbb{Z}^{\#\text{classes in } G}$

for $G$ a groupoid!

What makes this process even more interesting is the notion of "transfer" for (co)homology theories.

c) Generally, philosophically, homology is a covariant functor, and cohomology is a contravariant functor.

But sometimes we get the other sort of invariance too!

\[ \text{eg: } X \leftarrow M \rightarrow Y \rightarrow Z \text{ gives the usual } N_*: \text{Ho}(Y) \rightarrow \text{Ho}(Z) \]

but also the more interesting \[ M: \text{Ho}(X) \rightarrow \text{Ho}(Y) \]

d) Simple example involving transfer:

\[ \text{Diagram: } \]

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Then \[ H_0(5) \xrightarrow{M^4} H_0(7) \]

\[ aA + bB + cC \rightarrow (a+b)D + yE \]

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has the rather interesting "transfer" map \( M^4 \rightarrow (M^4)^7 \)

\[ \alpha D + \beta E \rightarrow \alpha(A+B) + BC \]

"Summing over the fiber"

- In fact, this is exactly how spans give rise to matrices. Both sides give us numbers.

- But the really interesting thing comes when we start with groupoids! We then turn them into topological spaces and thence into vector spaces.

  Spans turn into linear operators, and this happens by means of the "transfer trick."

  This is what we had called \textit{de-groupoidification}, although one can also call it the \textit{zeroth homology for groupoids}.

  This is the DECAT we use, as said above.