Though we will talk about representation theory—which involves groups and their actions on something—typically linear maps (invertible) on vector spaces—

nevertheless, since we'll deal with geometric map theory, we'll see more objects than just vector spaces.

e.g. $\text{Diff}(M) = \text{group of diffeomorphisms of a smooth manifold}$

preserve the differentiable structure.

Our guiding principle is:

\[
\text{Every transformation group is the group of "something-morphisms" for some essentially unique "structure".}
\]

What is structure? Very vaguely, it allows us to tell elements of a set apart. For example, in axiomatic theory (involving axioms about abstract predicates) structure on some sets, e.g. Euclidean geometry.

In some sense, symmetry and structure are dual in the same sense that intermediate fields are dual to subgroups: when one of them becomes bigger, the other grows smaller.

So we should see that there are two kinds of new about talking about structures: one is positive like actually putting an axiomatic system on some set...
(2) So we now seek a positive answer, or an axiomatic theory, to the following question:

(a) Given a transformation group $\mathcal{G} \subseteq S! = \text{Perm}(S)$, it

\[
\left[ \mathcal{G} = \text{group, } S = \text{set, action } \mathcal{G} \to S! \right]
\]

find the structure that is preserved by exactly $\mathcal{G}$

(from among $S!$)

(b) We'll work towards this goal. Thus, a transformation group $\mathcal{G}$ will give rise to an axiomatic theory.

In fact, the converse is also true and easier to do!

Given a complete (every statement is either true or false)

axiomatic theory, with an axiom stating that:

the "universe" of the model is bounded by $\forall\exists\forall\forall$:

Can we extract from it a

1. Every statement is either false or true and provable.
2) All models of the axiomatic system are isomorphic (e.g., the plane as a model of Euclidean geometry).

Given such an axiomatic system, take some model (which one?) and look at its automorphism group. This is a transformation group, and it acts on the universe, and the symmetries that it preserves are exactly an axiomatic system.

3) How does one go the other way? It's from a finitary transformation group (e.g., the symmetric group) to an axiomatic theory. Find an axiomatic theory!

Back to recovering the structure (positively) from the group of symmetries:

We'll introduce the orbit simplex picture of a group:

This is the orbit space of the action of the group on the simplex whose vertices are the elements of the set.

Example: $S = \{A, B, C, D\}$, $G = \text{permutations}$ that fix $A$. ($|G| = 2$).

We draw the simplex and draw the "pre" picture.
And having labelled the components, we now "fold" along all axes of symmetry obtained from G S S I! and look at this - this is the sub-simple picture. So, eg for each, the picture would be...