1. We continue with our comparison chart:

<table>
<thead>
<tr>
<th>Set Theory</th>
<th>Projective Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite sets</td>
<td>Finite dim vector spaces</td>
</tr>
<tr>
<td>( n )</td>
<td>( \mathbb{F}^n ) or ( \mathbb{F}P^{n-1} )</td>
</tr>
</tbody>
</table>

*\( (1- \text{dim subspaces}) \quad \text{(points)} *\

*\( \text{eg.: } n=3 \) *\

Here a "line" is just 2 pts

- Points, Points, Points, ...\n- Points, Lines, Planes, ...

\[ \text{D-flags} \quad \text{D-flags} \]

2. We now do our homework:

a) \( D = \begin{array}{c}
\text{i} = j = k = \cdots = n \\
\end{array} \)

Then \( \text{D-flags on } \mathbb{F}^n = \text{points of } \mathbb{F}P^{n-1} \)

(\( \text{D-flags on } X = \text{a set} \) = points of \( X \)!

\( \# | D(n) | = n \)

\( \mathbb{F}P^{n-1} = \mathbb{F}^n/\mathbb{F}^* \)

\( \mathbb{F}^* = \mathbb{F} - \{0\} \)

\( n = \frac{q^n - 1}{q - 1} \)

\( \Rightarrow | D(5) | = 5! / \binom{5}{3} \)

\( = \frac{5!}{2!3!} \)

\( 96 \text{ ways to look for } \binom{5}{3} = \text{sets} \)

\( \Rightarrow \binom{5}{3} = 10 \)
This is often called the Young subgroup, these are the perm's precisely preserving an $O^-$ flag. 

$(S_5, S_5)$ acts transitively on all $D$-flags, and $\text{denom.} = \text{stab}(p^0) = S_5$. 

We'll now try to count here using the same philosophy. 

Say we want a transitive $O^-$ action and then find stabilizers. 

The key to go from $S_5$ to $\text{GL}(5,F)$ which acts transitively on $D(F^5)$ for sure. 

Then it's enough to show/find stabilizer subgp of $\binom{\star}{2}$. 

But this is $\binom{\text{GL}_2}{0, \text{GL}_3} \rightarrow \text{which is an example of}$ 

A parabolic subgroup of $GL_n$ is one that preserves some $D$-flag on $F^n$. 

The key now is to count various subgps: 

$$|D(F^5)| = \frac{\left| \text{GL}(5,F) \right|}{\left| \left( \text{GL}_2(F) \times \text{GL}(3,F) \right) \times F^{2 \times 3} \right|}$$ 

$(A,B)$ acts on map $F^2 \rightarrow F^3$ via: 

$(A,B) \circ \phi = B \circ (A \cdot \phi)$ etc. 

Remark: The reason is put in $x$ here is that eventually we will want to categorify everything, and want both sides to be mon. in some category. We want the "simplest possible $x$"!
So - what's $|GL(n, F_q)|$?

Ans: Look at it column by column. The first column is

... so $\mathbb{F}_{q^{n-1}}$.

The second is anything not from the line so $q^{n-1}$

do...

$|GL(n, F_q)| = \prod \left( F_q - \{0, 1\} \right) x \left( F_q^n - F_q^{n-1} \right) x \left( F_q^n - F_q^{n+1} \right) \cdots$

\[ = (q^{n-1})(q^{n-2}) \cdots (q^n - q^{-1}) \]

\[ = \cdots = [n]_q! \cdot (q-1)^n q^{\binom{n}{2}} \]

So what we now do is

$|D(F_q^5)| = \frac{\binom{5}{1} (q-1)^5 q^5}{[2]_q! \cdot (q-1)^2 q^2 \cdot [3]_q! (q-1)^3 q^3 \cdot q^5}$

This becomes a product of 3 factors:

1. The $q$-binomial coefficient $\frac{\binom{5}{1}}{[2]_q! [3]_q! q}$

2. $(q-1)^5$ cancel out

3. $q$ also cancel: $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + (m \cdot n)$

because in pictures.
This happens in general too: For $D = \begin{bmatrix} \ddots \end{bmatrix}$ we have $D(F^n) = \text{Grassmann of } l \text{-dim subspace of } F^n$.

$= \text{Sym}(F_l)$ which we now call \( \begin{bmatrix} (n) \\ 1 \end{bmatrix} \).

Then one can do this in general:

for sets and for vector spaces (see the analysis of an earlier class).

So we end up with

\[
\begin{bmatrix} n \\ n_1 \ldots n_k \end{bmatrix}_F \quad \text{and} \quad (F = F_q) = \sqrt{1}.
\]

\[4\]

In connection with above follow:

\[\text{Defn.} \quad \text{For any } n \text{-box } \text{Unrammed } Y(D, D(F^n)) \text{ is denoted by } \begin{bmatrix} n \\ n_1 \ldots n_k \end{bmatrix}_F.\]

Remark 1: Just like \( \begin{bmatrix} n \\ n_1 \ldots n_k \end{bmatrix}_F \) is in \( \mathbb{Z} \) and not just \( \mathbb{Q} \),

why are \( \begin{bmatrix} n \\ n_1 \ldots n_k \end{bmatrix}_F \) in \( \mathbb{Z}[q] \), and not just \( \mathbb{Q}(q) \)?

Remark 2: As \( \begin{bmatrix} n \\ n_1 \ldots n_k \end{bmatrix}_F \) is symmetric in \( n \)'s, is there a bijection

\[b/w \text{ say } \begin{bmatrix} 5 \\ 33 \end{bmatrix}_q \leftrightarrow \begin{bmatrix} 5 \\ 32 \end{bmatrix}_q?\]

Remark 5: More gleaning of information/observations about \( \begin{bmatrix} \cdots \end{bmatrix}_q \): \( [n]_q = 1 + q + q^2 + \cdots \).
Schubert

\[ F_{p^{n+1}} = \prod_{k \geq 0} \text{cell decomposition} \ 1 + F + F^2 + \ldots \]


Why does \[ \text{Remark 1} \] hold? It comes from cohomology of Schubert cells.

Says Grassmann, since every multilinear form is in the cohomology.

So, look at \[ \binom{n}{d} \]. Then \[ f_p(n, d) = \# \text{Schubert cells} \] and \[ \# \text{cells of degree } d \] gives

\[ \sum_{i=1}^{\infty} \eta_i q^d = 1 \]

Hence \[ \binom{n}{d} \in \mathbb{Z}_{\geq 0} \ [q] \], not just \([q] \). \]

Example \( \binom{4}{2} = \#2\text{-planes in } F^4 \)

\[ \binom{4}{2} = \binom{4}{2}_p = q^0 + q^1 + 2q^2 + q^3 + q^4 \]

Speaking projectively, \# lines in 3-(proj)-space

Thus, \[ q^0 = \text{most special family} \]
\[ q^1 = \text{slightly less special} \]
\[ q^2 = \text{slightly less} \]
\[ q^3 = \text{most general, hence } F^4 \text{-mary} \]
In fact \( q^3 \rightarrow \text{our line is the line} \)

\( q^4 \rightarrow \text{our line contains pt. } \text{and NO BETTER} \)

\( q^5 \rightarrow \text{our line is plane} \)

\( q^6 \rightarrow \text{our line contains the point} \)

\( q^7 \rightarrow \text{our line intersects the plane line} \) (not at the point)

\( q^8 \rightarrow \text{and no better!} \)

\( q^9 \rightarrow \text{etc.} \)

\( q^{10} \rightarrow \text{Remark 12 will be addressed later on in the course...} \)

\( \text{NOTE} \) Both Remarks only need to be explained for \( q \)-binomial coefficients, because every multinomial coeff is a product of binomial coeffs.

Moreover, this should transfer over to the picture, inside of sets, no some sort of functionality.