"Ask TB for explanations of the 3 "Remarks" from last lecture (see notes!)."

4. As remarked in an earlier lecture, the idea here is to try and categorify integers, so that a formula e.g. a summation, is converted into e.g. a set being identified as being partitioned into some "special" subsets.

E.g. Fix an uncombed Young diagram \( D(m) \).

Then \( |D(n)| = \binom{n}{m_{1}, \ldots, m_{k}} \in \mathbb{N} \quad \leftarrow \quad |D(F_{q})| = \binom{n}{m_{1}, \ldots, m_{k}}_{q} \quad \lim_{q \to 1} \downarrow \quad 1 \cdot 1 \)

Decategorification

\[ \text{"link"} \quad \frac{q \to 1}{\text{Product}} \]

Several things are missing from this picture:

@ Is there a valid map, where one does a "base change"?

@ Is there such a map so that the above picture commutes?

@ Does it even make sense to speak of a commuting picture here?

@ Even more basic: What families do these belong to?
Kelly \( (n, -mk) \in \mathbb{N} \)
\[
\begin{align*}
\left( \frac{n}{m} \right)_q & \in \mathbb{Z} \quad (q) \\
D(F_q^n) & \in \mathbb{N} \quad \text{Boij-Verdier}
\end{align*}
\]

But this picture is actually better than it looks! Because

**Lemma** \( \left( \frac{n}{m} \right)_q \in \mathbb{Z}_{\geq 0} \quad (q) \)

The justification behind this answers one question said last week (Remark 0)

\[ \rightarrow \text{ uses Schubert cells} \]

and **Note**: because \((\text{multiset}) = 11\) dimensionally

Hence it suffices to show that \( \left[ \frac{n}{k} \right]_q \in \mathbb{Z}_{\geq 0} \)

\[ (n, -mk) \in \mathbb{N} \]

1. "Justification" - easy example: \( D = \mathbb{H} \) \( n = 3 \)
\[
\Rightarrow D(F^3) = \mathbb{F}P^2 \quad \text{lines in } F^3 \quad (\text{points in } \mathbb{F}P^2)
\]

Then \( D(F^3) = F^3 \setminus \mathbb{S}_0^2 = 1 + F + F^2 \). How?

\[ \mathbb{F} \setminus \mathbb{S}_0 \]

\[ \text{Recall} \quad \mathbb{F}P^{n-1} = \mathbb{F}P^{n-2} \sqcup \mathbb{F}P^{n-1} \quad \text{because this is the hyperplane @ } \mathbb{P}^n \]

\[ \text{e.g. lines in } F^2 = F \quad \text{the plane @ } \mathbb{P}^2 \]
So now what we're basically doing is:

\[ D_0 = \begin{array}{c} \text{total flag} \end{array} \]

Pick a particular total flag \( D_0 = \square \)

A D-flag could relate to this via:

\[ \begin{array}{c}
\begin{cases}
\circ & \circ = \emptyset \\
\circ, & \circ = F^1 \\
\circ, & \circ = F^2
\end{cases}
\end{array} \]

This gives us a decomposition of \( F^2 \) into three Borel classes \( F^-, F^0, F^+ \).

(2) Borel classes in general

(2) A flag variety \( D(F^n) \) is the disjoint union of Borel classes, obtained as follows:

Write \( D_0 = \begin{array}{c} \text{total flag} \end{array} \)

and now we look at how a D-flag \( D_0 \)-flag \( \begin{array}{c} \text{total flag} \end{array} \)

Choose a \( D \)-flag that intersects intersects with this fixed \( D_0 \)-flag \( \begin{array}{c} \text{total flag} \end{array} \).

(6) This is why Jim introduced the lemma last time:

\[ \text{Free(Orbit}(X \times Y)) \cong \text{Hom}_G(\text{Free}(X), \text{Free}(Y)) \]

Here, \( G = GL(F^n) \).

Now write \( D(F^n) \times D_0(F^n) = \bigsqcup B_r \), as a disjoint union of \( G \)-orbits.
Next, write \( D(F^*) = \{ i \} \) for \( (\Phi, \Phi) \in B_i \). \( \Phi = D_0 = \text{total flag} \)

and these components are \( B_i \) classes.

0. The Schubert cells are the closures of \( \zeta \) and hence are (disjoint) unions of \( B_i \) classes.

\[ \text{eg. } D = \boxed{\square}, D_0 = \boxed{\square}, \zeta = (P_0, L_0, T_0), \]
\[ D = (L) \]

(i) \[ \text{P}_0 \leq L \leq T_0 \quad \text{(and no better)} \quad F \]

\[ \text{F}^2 (\text{slope = f, intercept = f}) \]

(ii) \[ L \leq T_0 \quad \text{(and)} \]
\[ \text{F}^2 (\text{slope = f, intercept = f}) \]

(iii) \[ \text{P}_0 \leq L \quad \text{(and)} \]
\[ \text{F}^2 (\text{slope = f, intercept = f}) \]

(iv) \[ \exists P' \quad \text{P}' \leq L \quad \text{(and)} \]
\[ \text{F}^3 (\text{F}^2 \text{slope, F} = \text{intercept}) \]

(v) \[ (a, n, b, \alpha) \]
\[ \text{F}^4 (\text{F}^2 \text{slope, F} = \text{intercept}) \]
\[ \text{(4)} \quad D(F^4) \equiv 1 + F + F^2 + F^3 + F^4 \]

\[ \Rightarrow D(F^4) = \binom{4}{2} = 1 + q + 2q^2 + 4q^3 + q^4 \]

That it is an \( N[ q ] \) is not obvious from the definition. But we can explicitly compute it in base \( q \) and check that it agrees with above:

\[ \binom{4}{2} = \underbrace{1.111111} \_ = \underbrace{1111.1111} \_ = \overbrace{11211} \]

\[ = q^4 + 2q^2 + q + 1 \]

\( \text{(3)} \) We'll see that \( \binom{n}{k} \in N[q] \) using Benford classes.

But first, we show that \( \binom{n}{k} \) is palindromic

(because then every \( \binom{n}{n-k} \) is too!)

**Proof** To see they are has to replace \( q \) by \( q^{-1} \) and see if one gets the same thing (modulo a power of \( q \)).

\[ \frac{(q-1)(q-1)}{(q-1)(q-1)} \Rightarrow \frac{q^{-1} \cdot 1}{q^{-1} \cdot 1} \]

\[ = (1-q^{-n})(1-q^{-k}) \]

\[ = \frac{(q^{-1} \cdot q^{-1}) \cdot (q^{-1} \cdot q^{-1})}{(q^{-1} \cdot q^{-1}) \cdot (q^{-1} \cdot q^{-1})} \]

\[ = \frac{(q^{-n} \cdot q^{-k}) \cdot (q^{-n} \cdot q^{-k})}{(q^{-n} \cdot q^{-k}) \cdot (q^{-n} \cdot q^{-k})} \]
4) How to show each \( \binom{n}{k} \) in \( \mathbb{Z}_q \)?

\( \text{Reduced} \)

One can actually do \( \mathbf{2} \) \( \mathbf{0} \) using Reduced Echelon Form!

and this kind of thing would work in general!

This gives a Bézout class decomposition for any Gaussian.

This "proves" that \( \binom{n}{k} \in \mathbb{Z}_q \).

This proves it for \( \binom{n}{n_i \rightarrow n_k} \), as well!

(FAILED APPROACH)

6) Alter Proof the following "Pascal-Type Formula":

\[
\binom{n+k}{k}_q = -q \binom{k-1}{q} \binom{n}{k}_q + \binom{n-k+2}{k-1}_q \binom{n}{k-1}_q
\]

Pf: Expand the RHS to get:

\[
\frac{-q \binom{k-1}{q} \binom{n}{k}_q + \binom{n-k+2}{k-1}_q \binom{n}{k-1}_q}{\binom{k}{q} \binom{n+k}{q} \binom{k}{q} \binom{n-k}{q}}
\]

= \[
\frac{\binom{n+k}{k}_q}{\binom{k+k}{k}_q} \left( -q \binom{k-1}{q} \binom{n-k+1}{k}_q + \binom{n-k+2}{k-1}_q \binom{k}{q} \right)
\]
and the term in the parentheses is

\[- q \left( \frac{q^{n+1} - (q^{n-k+1} - 1) + (q^{n+2} - q^{n-k+2})}{(q-1)^2} \right)\]

\[= -q \left( \frac{q^n - q - q^{n-k+1} + 1 + q^{n+2} - q^{n-k+2}}{(q-1)^2} \right)\]

\[= q^{n+2} - q^n - q - q^{n+1} = \frac{(q^{n+1} - 1)(q^1)}{(q-1)^2} = \frac{[n+1]_q}{q-1}\]

Hence the entire RHS now becomes

\[\frac{[n]_q}{[k]_q} \cdot \frac{[n+1]_q}{[n-k+1]_q} = \binom{n+1}{k}_q\]

\[\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q\]

\[\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q\]

\[\text{Let's check this! Start as usual with the RHS.}\]
\[ q^k \binom{n-1}{k}_q + (n-1)_q = \frac{(n-1)!}{(k)_q! \cdot [n-k]_q!} \left( q^k [n-k]_q + [k]_q \right) \]

\[ = q^k \binom{n-k}{k-1}_q + (q^k - 1) = q^k \binom{n}{k}_q - 1 = [n]_q \]

\[ \Rightarrow \text{RHS} = \frac{[n-1]_q}{[k]_q! \cdot [n-k]_q!} \]

\[ = \binom{n-1}{k}_q \]

So once this claim is done, one uses the "base case"
\[ \binom{n}{0}_q = 1 \quad \forall \ n > 0 \]

to show that the \( \binom{n}{k}_q \)'s are, indeed, in \( \mathbb{N}[q] \)
by induction on \( n, k \).

(Proved).

(And the categorical geometric meaning is, as was said today, in terms of Bruhat classes and Schubert cells.)

\[ \text{Note: That the } q \text{- binomials are palindromic, comes from thinking about these } \binom{n}{k}_q \text{ as Hilbert polynomials of compact real manifolds (i.e. if their cohomology).} \]

Then, that these polynomials are symmetric/palindromic is but what we better know as Poincaré duality!