Today's theme: Use Hecke operators as a tool for understanding Representation Theory.

Example: $G = \text{finite group}$, $R = \text{doubly transitive permutation rep. of } G$ with $G = G^G$ and $M$ a $G$-rep.

Lemma: Then $R = G \oplus M$ as $G$-modules, with $G = G^G$ and $M$ a $G$-rep.

Definition: $R = \text{perm rep of } G$ means $\exists \text{ set } S$ s.t. $R = \text{Free}(S), \ G \leq S! \ (|S| > 1)$.

$D$: $\text{Doubly transitive perm rep } \iff G \text{ acts on } S^2 \equiv \exists ! 2\text{-orbits } \Delta_5 \ (S^2 \setminus \Delta_5)$.

$G$ acts transitively on each of these.

The proof uses Hecke operators, and Schur's Lemma as follows:

"The reps of a finite group $G$, form an orthonormal basis for the $2$-Hilbert space of finite dim $G$-reps."

One attaches formal coeffs to reps, and then makes $(G\text{-rep})^{\mathbb{C}}$ into a ring. For inner products, one uses

$\langle V, W \rangle := \text{Hom}(V, W) \cdot (\text{Span Set of Hecke}) = \text{a vector space of operators}$.
One can extend so that linearity works here.

This makes the category nice enough to be made into a $\mathbb{C}$-Hilbert space. But because it seems to be much more — namely, a categorification of some sort! — we call it a $2$-Hilbert space.

But then the maps behave nicely w.r.t. $\text{Hom}$, i.e.
\[
\dim \text{Hom}(V, V') = \delta_{V, V'} \quad (V, V' \in G \text{- reps})
\]

and this is what Schur's Lemma says.

\[\text{Note: The use of the word "basis" is because of Maschke's Theorem (which says: $G$-rep is a semisimple category).}\]

\[\Box\]

2. Back to our example. This is immediate from the previous class:

$\text{Hom}_G(\mathbb{C} \mathbb{C})$ has a basis given by the $G$-orbits $w_i$. But by double transitivity, this # = 2

So $\dim \text{Hom}_G(\mathbb{C} \mathbb{C}) = 2$. By Schur+Maschke, if $R = \bigoplus_{i \in \mathbb{C}} m_i V_i$, then $\langle R, R \rangle = \sum_{i \in \mathbb{C}} m_i^2$

So it must be $1^2 + 1^2$, i.e. $\sum_{i \in \mathbb{C}} m_i = 1$.
Moreover, the diagonal matrix $A \preceq S^2$ will correspond to the trivial representation if we use the proof of the result of that previous class.

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We'll do a more powerful application now: we'll "find" or "characterise" all the groups of (4I).

By the above lemma, we're trying to find some basis $\{v_i\}$ of a Hilbert space. We'll use (the categorified version of) "Gram-Schmidt Orthonormalization".

So, we need to start with some basis and go on.

\textbf{Miracle} Why this categorified version? Because then we work with actual objects in our category, and (hopefully) end up with actual objects in our orthormal basis.

Note that in any old Hilbert space, there are only many orthonormal bases. But in our setup, we only have one basis that also consists of objects in our category! (Called a "categorical basis").

So, a priori we should expect our use of the Gram-Schmidt process to yield a categorical basis. But, miraculously, it does!

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Back to our example, we look at the flag $\mathcal{F}$ of $S^4$ (defined earlier). These are given (by 'ly) by...
There are not direct, but an example class, we can find a cycle (which are also indexed by these Young tableaux) inside the corresponding flag-snap.

Moreover, we start with this basis and carry out from Schmidt. How did we know the above basis was a basis?

Easy answer: There isn't, but go ahead anyways! If we end up with non-zero entries at the end, then we know we're good! (And it will be thus!)

(Actual answer: needs math...)

But first, let's write down the $\langle, \rangle$ matrix:

And the matrix is symmetric! In general b/c

$\text{Hom}_G(V, W) \cong \text{Hom}_G(W^*, V^*)$

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So dim's are the same!