The classical and quantum Pascal's Triangle

Write the coefficients of $x^k y^{n-k}$ in the $n$th row.

$n = 0$ \quad 1

$n = 1$ \quad 1 \quad 1

$n = 2$ \quad 1 \quad 2 \quad 1

$n = 3$ \quad 1 \quad 3 \quad 3 \quad 1

And this shows us a recursion relation that the \( \binom{n}{k} \)'s satisfy:

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

(b) But of course, we'll try to categorify this. So:

LHS: \( k \)-elt subsets of \( n \)

RHS: \( (2 - \text{shrink}) = \frac{1}{2} \binom{n}{k} \) \( k \)-elt set \( \Rightarrow \) \( \binom{n-1}{k} \)

\# \( k \)-elt set \( \Rightarrow \) \( \binom{n-1}{k} \)

(we pick a distinguished elt. \( 1 \in [n] \) )

(c) Now, let's $q$-deform all this. How does one get a relation between quantum binomial coefficients?

One can compute, and see that \( \binom{n}{k}_q \) does NOT equal \( \binom{n-1}{k}_q + \binom{n-1}{k-1}_q \) (so \( 0 < k < n \)).

Here's how to categorify: So \( \binom{n}{k}_q = \# k \text{-dim subspaces} \)
Let's in fact work over a general field $F$.

So, to use $(n-1)$ here we fix a $(n-1)$-dimensional subspace $S_{n-1}$ and consider our $k$-subspace $S_k$ in relation to this hyperspace.

One option is that $S_k = S_{n-1}$. Otherwise, we claim that $S_k \not\subseteq S_{n-1}$ has dimension $k-1$. Because:

$$S_k \cap F^n$$

has dimension $k$, and $S_k \cap S_{n-1} \neq k$, so

$$S_k \cap S_{n-1} = (k-1) - \dim.$$  

And how many such degrees of freedom for the extra degree of freedom ($\dim S_k / S_k \cap S_{n-1}$) are there?

It can be from anything as the remaining $n-k = (n-1) - (k-1)$ degrees of freedom.

So,

$$\binom{n}{k}_F = \binom{n-1}{k}_F + \binom{n-k}{k-1}_F$$

as in the category of sets.

(\text{De-categorifying: } F \rightarrow \mathbb{F}_q) \text{ so we now get the } q\text{-Pascal identity:}

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-k}{k-1}_q$$

One now writes the $q$-Pascal triangle:
Observations:

1. Coeff's are \( N_q \), 
   of the 
   \[ q \text{-Pascal identity} \]
   (This is the other proof, not using Row Echelon Form)

2. There are two symmetries involved:

   a. Each entry is palindromic (which can be shown by replacing \( q \) by \( q^{-1} \), etc.)
      Pommerscheidt, in the setup of Grassmanns
      and Brouet/Schubert cells!

b. The entire picture is symmetric in each
   row is "palindromic," which will
   lead to a proof that \( q \)-biomial and \( q \)-multinom
   Coeffs are "symmetric" in their arguments.
   We'll see this next time!
3) Just as any row in the classical Pascal $\triangle$ adds up to $2^n$, or if we attach $2^{n-k}$ to $(x+y)^n$, what is the $q$-analog to the $q$-binomial theorem?

This is the question that will occupy us for the rest of the lecture.

2) If we drop a ball from the top, and it chooses left/right at each stage with probability $\frac{q}{2}$, then we get a "Gaussian" like below at the base.

More specifically, \( (n \choose k) \)  the $\#$ ways to get from \((0,0)\) to \((n,k)\) = \((n \choose k)\)

\[ n \choose k \]

\( \text{Pf:} \) From \((0,0)\), have to go left $n-k$ steps and right $k$ steps, and we can do either at each stage.

So $\checkmark$

5) How does this $q$-generalize? Say we're looking at paths again. Then as height each path, by not just a power of $q!$, we get for \((4)\):

<table>
<thead>
<tr>
<th>Path</th>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>$q$</td>
</tr>
<tr>
<td></td>
<td>$q^2$</td>
</tr>
<tr>
<td></td>
<td>$q^3$</td>
</tr>
<tr>
<td></td>
<td>$q^4$</td>
</tr>
</tbody>
</table>
Other than this complicated method of multiplying all "numbers written on lines" to get the weight, what's a more "seeable" method?

Well, each such path determines an area to the right! e.g.

\[ 9 \times 9 = 2 \text{ units/block} \]

Hence weight = $9^3$.

Let's make this completely specified now.

Clearly, for \( n \times k \), we have to work with all possible paths in an \( n \times k \) rectangle.

And now every path (weighted) creates a Combed Young diagram! Which one?

**Defn:** Let \( \Omega_{nk} = \) \# Combed Young diagrams with \( \leq k \) columns & \( \leq n - k \) rows

eg. \[ \begin{array}{ccc} & & \hline & \hline & & \end{array} \]

etc.
Then \( |\Omega_{\eta,k}| = \binom{n}{k} \), and

\[
\binom{n}{k} = \sum_{\omega \in \Omega_{\eta,k}} q^{|\omega|}, \quad \text{where} \ |\omega| = \# \text{boxes in } \omega
\]

Another approach is to use Bézout cells and \( \emptyset \) with our pre-fixed Complete flag \( F_0 \) of \( F_p \)

We have any \( F_0 \) is not \( \emptyset \), and \( F_0 \). Eventually, \( F_0 \) reaches \( F_p \), so \( F_0 \) \( \emptyset \) \( F_p \) reaches \( F_p \)

And we're seeing when the dimensions start to increase \( \Delta \) dim by \( 1 \)

Moreover, the \( \eta \) you start out with (from the chosen path) is exactly the shape that \( \Phi \) shows up in the associated real echelon form as the undetermined \( k \)'s in it!