Higher-Dimensional Algebra VII: Groupoidification

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DRAFT VERSION ONLY

This is a very rough draft, which only attempts to lead up to a precise statement of the Fundamental Theorem of Hecke Operators Theorem 18. We include neither proofs, which will eventually be provided, nor motivation — which will also be provided, but for now can be found online in the reams of material associated to the Fall 2007 seminar on Geometric Representation Theory [2].

1 Degroupoidification

The idea of groupoidification is to take interesting pieces of linear algebra, such as the representation theory of finite groups, and systematically replace vector spaces by groupoids and linear operators by spans of groupoids. Equations between linear operators then become interesting maps between spans of groupoids.

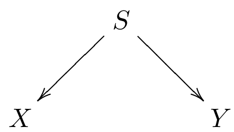
So, groupoidification is a special form of categorification. As always, the hope is that we are not just making stuff up, but *revealing* structures in mathematics that were *already implicitly there*. And, as always, this is not a systematic process — but its reverse *is* a systematic process. We call this 'degroupoidification'.

Degroupoidification is a process going from spans of groupoids to linear operators between vector spaces. To avoid issues of analysis we limit ourselves here to spans of finite groupoids.

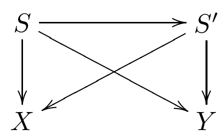
Lemma 1. *There is a weak 2-category* FinSpan *where:*

- *the objects are finite groupoids,*

- *a morphism* $S\colon X \nrightarrow Y$ *is a span of finite groupoids*



- *a 2-morphism* $f\colon S \Rightarrow S'$ *is a diagram*



  commuting up to natural isomorphism,

- *composition of morphisms is done via weak pullback of spans,*

- *composition of 2-morphisms is done in the obvious way,*

- *all identity morphisms and 2-morphisms, left and right unit laws, and associators are the obvious things.*

(Note: To obtain a bicategory FinSpan, we must arbitrarily choose a weak pullback for each composable pair of spans. A more principled — but ultimately equivalent — approach would avoid these choices and yield, for example, an opetopic bicategory. We prefer to use traditional bicategories because they are more familiar. So, henceforth we use 'weak 2-category' as a synonym for 'bicategory')

Next, we fix a field k of characteristic zero. Let FinVect be the category whose objects are finite-dimensional vector spaces and whose morphisms are linear operators. We shall often think of this as a weak 2-category with only identity 2-morphisms.

Our goal is now to describe a weak 2-functor

$$D\colon \mathrm{FinSpan} \to \mathrm{FinVect}$$

called 'degroupoidification'. For this we need a bit of information about the zeroth homology of groupoids. This can be seen either as a covariant functor, via 'pushforward', or as a contravariant one, via 'transfer'. (The name 'transfer' is traditional in the homology theory of spaces; sufficiently nice maps between spaces give not only a pushforward in homology but also a 'transfer' going back.)

Let FinGpd be the weak 2-category of finite groupoids, functors and natural isomorphisms. Given $X \in \mathrm{FinGpd}$, we define its **zeroth homology** with coefficients in k, $H_0(X)$, to be the free vector space on the set of isomorphism classes of objects of X. Given a functor $f\colon X \to Y$ between finite groupoids, we define its **pushforward**

$$f_*\colon H_0(X) \to H_0(Y)$$

by

$$f_*[x] = [f(x)]$$

where $[x]$ is the isomorphism class of $x \in X$. It is easy to check that naturally isomorphic functors give the same pushforward, and:

Lemma 2. *There is a weak 2-functor from* FinGpd *to* FinVect *sending:*

- $X \in \mathrm{FinGpd}$ *to its zeroth homology* $H_0(X) \in \mathrm{FinVect}$,
- $f\colon X \to Y$ *to its pushforward* $f_*\colon H_0(X) \to H_0(Y)$,
- $\alpha\colon f \Rightarrow g$ *to the identity.*

Given a functor $f\colon X \to Y$ between finite groupoids, we can also define a linear operator

$$f^!\colon H_0(Y) \to H_0(X)$$

called its **transfer**. This maps each isomorphism class $[y]$ of objects in Y to a cleverly weighted sum over objects in X that map to it — or more precisely, to objects isomorphic to it. The 'clever weighting' involves the concept of groupoid cardinality [1], but we will supress that concept here for the sake of efficiency. We still need a little notation:

Definition 3. *Given a category X, let $\underline{X}$ be the set of isomorphism classes of objects of X.*

Definition 4. *Given a category X and an object $x \in X$, let* $\mathrm{aut}(x)$ *be the group of automorphisms of x.*

Definition 5. *Given a functor $f\colon X \to Y$ and an object $y \in Y$, let $f^{-1}(y)$ be the* **essential preimage** *of y, that is, the full subcategory of X consisting of all objects that f maps to objects isomorphic to y.*

We define the transfer by:

$$f^![y] = |\mathrm{aut}(y)| \sum_{[x] \in \underline{f^{-1}(y)}} \frac{[x]}{|\mathrm{aut}(x)|}$$

Maybe it's better to say this in words too. To compute $f^![y]$, we sum over objects $x \in X$ mapping to objects isomorphic to y, picking one object x from each isomorphism class. This sum is cleverly weighted by factors involving the size of various automorphism groups. I hope I have these factors right! The first way to check it is to prove this lemma:

Lemma 6. *There is a contravariant weak 2-functor from* FinGpd *to* FinVect *sending:*

- $X \in \mathrm{FinGpd}$ *to its zeroth homology* $H_0(X) \in \mathrm{FinVect}$,
- $f\colon X \to Y$ *to its transfer* $f^!\colon H_0(Y) \to H_0(X)$,

- $\alpha\colon f \Rightarrow g$ *to the identity.*

Now for the punchline:

Proposition 7. *There is a weak 2-functor,* **degroupoidification***,*

$$D\colon \mathrm{FinSpan} \to \mathrm{FinVect}$$

sending:

- *any object* $X \in \mathrm{FinSpan}$ *to its zeroth homology* $H_0(X) \in \mathrm{FinVect}$*,*
- *any morphism*

$$\begin{array}{ccc} & S & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle q} \\ X & & Y \end{array}$$

  to the operator $q_* p^! \colon H_0(X) \to H_0(Y)$*,*
- *any 2-morphism* $\alpha\colon S \Rightarrow S'$ *to the identity.*

Note that since FinVect is a weak 2-category with only identity morphisms, this is all the information we need to provide to describe a weak 2-functor. The main job in proving the lemma is showing that composite spans get sent to composite linear operators. This is where the 'clever weighting' becomes important: the lemma works when you get this weighting right.

2 Nice Topoi

It is profitable to study groups, or more generally groupoids, by their actions on sets. In fact, any groupoid has a topos of actions, which serves as a kind of stand-in for the groupoid itself. So, the degroupoidification process we described above can also be thought of as a process sending such a topos to a vector space.

As in the previous section, we will impose finiteness restrictions to avoid some technicalities that complicate things. If X is a finite groupoid, there is a category Set^X whose objects are actions of X on sets — that is, functors

$$A\colon X \to \mathrm{Set}$$

This category is a topos of some very nice sort:

Definition 8. *A category is a* **nice topos** *if it is equivalent, as a category, to* Set^X *for some finite groupoid* X*.*

Note: the reader does not need to know what a 'topos' is to make sense of the above definition. We would naturally like an intrinsic characterization of nice topoi.

A functor $f\colon X \to Y$ between finite groupoids induces a 'pullback' map

$$f^*\colon \mathrm{Set}^Y \to \mathrm{Set}^X$$

but this has a left adjoint

$$f_*\colon \mathrm{Set}^X \to \mathrm{Set}^Y$$

So, any span of finite groupoids

$$\begin{array}{ccc} & S & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & Y \end{array}$$

gives rise to a functor

$$q_* p^*\colon \mathrm{Set}^X \to \mathrm{Set}^Y$$

Definition 9. *A functor between nice topoi is* **nice** *if it is naturally isomorphic to one of the form*

$$q_* p^*\colon \mathrm{Set}^X \to \mathrm{Set}^Y$$

for some span of finite groupoids

$$\begin{array}{ccc} & S & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & Y \end{array}$$

Definition 10. *A natural isomorphism between nice functors is* **nice** *if it is equal to one arising from an equivalence of spans of finite groupoids.*

We would also like intrinsic characterizations of nice functors and nice natural transformations.

There is a 2-category Nice of nice topoi, nice functors and nice natural isomorphisms, and:

Lemma 11. *There is an equivalence of weak 2-categories*

$$J\colon \mathrm{FinSpan} \to \mathrm{Nice}$$

sending each finite groupoid X to the nice topos Set^X.

Combining this and Proposition 7, we get a weak 2-functor from Nice to FinVect. Namely, we choose a weak inverse to J, say

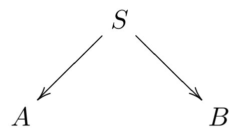
$$K\colon \mathrm{Nice} \to \mathrm{FinSpan}$$

This gives an alternative form of degroupoidification,

$$DK\colon \mathrm{Nice} \to \mathrm{FinVect}.$$

3 Hecke Theory

We now apply degroupoidification to an important example. Suppose G is a finite group. Given G-sets A and B, there is an obvious notion of a morphism from A to B, namely a G-invariant map $f\colon A \to B$. However, it turns out to be much more interesting to consider a different sort of morphism, namely a span of G-sets



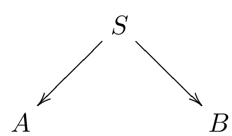
where the arrows are G-equivariant maps. But in fact, such spans are the objects of a nice topos!

To see this, recall the concept of 'action groupoid' or 'weak quotient':

Definition 12. *Given a set S with an action of a group G on it, the* **weak quotient** *$S//G$ is the groupoid with elements of S as objects, and pairs $(g, s) \in G \times S$ as morphisms from s to gs, where the composite of $(g, s)\colon s \to s'$ and $(g', s')\colon s' \to s''$ is defined to be $(g'g, s)\colon s \to s'$.*

Lemma 13. *Given a set S with an action of a group G on it, the category of G-sets over S is equivalent to the category $\mathrm{Set}^{S//G}$. When G and S are finite, so is $S//G$, so $\mathrm{Set}^{S//G}$ is a nice topos.*

So, given finite G-sets A and B, we can think of a span of G-sets

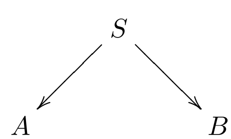


as a G-set over $A \times B$. By the above lemma, this is an object of the nice topos $\mathrm{Set}^{(A\times B)//G}$.

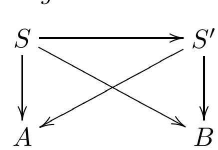
So, unravelling what we've done, we see:

Lemma 14. *Suppose G is a finite group and let A and B be finite G-sets. Then there is a nice topos $\mathrm{Set}^{(A\times B)//G}$ where:*

- *objects are spans of G-sets*



- *morphisms are commuting diagrams*



As mentioned, we want to think of a span of G-sets from A to B as a kind of morphism from A to B. The morphisms mentioned in the above lemma should thus be 2-morphisms between these! We can make this more precise:

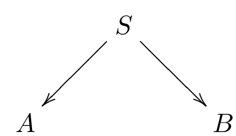
Lemma 15. *For any finite group G there is a weak 2-category enriched over* Nice*, the* **Hecke bicategory** *of G, denoted* $\mathrm{Hecke}(G)$*, where:*

- *objects are finite G-sets,*
- *given finite G-sets A and B,* $\hom(A, B)$ *is the nice topos* $((A \times B)//G)^{\mathrm{Set}}$.

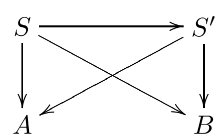
This is not a very precise statement of the lemma, since we haven't said how composition and the like are supposed to work! We also need to say how Nice is a *monoidal* bicategory, in order to speak of bicategories enriched over it. But for now, let us combine Lemmas 14 and 15, just to see where we stand:

Lemma 16. *For any finite group G, the weak 2-category* $\mathrm{Hecke}(G)$ *has:*

- *finite G-sets as objects*
- *spans of G-sets as morphisms*



- *commuting diagrams*



as 2-morphisms.

This should make it clearer how composition works: we use the usual composition of spans, via pullback. Indeed, it should be pretty obvious how $\mathrm{Hecke}(G)$ is a bicategory. So, the 'tricky part' is just showing that $\mathrm{Hecke}(G)$ is enriched over Nice.

But now let us plunge ahead and state the Fundamental Theorem of Hecke Algebras! For this, we first recall that by Lemma 11, the weak 2-functor

$$J \colon \mathrm{FinSpan} \to \mathrm{Nice}$$

has a weak inverse

$$K \colon \mathrm{Nice} \to \mathrm{FinSpan}$$

In fact J is monoidal (with respect to some obvious monoidal structures on FinSpan and Nice), and so therefore is K. This allows us to do 'base change' and convert any bicategory B enriched over Nice into a bicategory $\overline{K}(B)$ enriched over FinSpan. Here we use an overline to denote base change with respect to some weak monoidal 2-functor.

Similarly, degroupoidification

$$D\colon \mathrm{FinSpan} \to \mathrm{FinVect}$$

is monoidal. So, we can convert any bicategory B enriched over FinSpan into a bicategory $\overline{D}(B)$ enriched over FinVect.

We can do both sorts of base change starting with the Hecke bicategory of G: Hecke(G) is enriched over Nice, so $\overline{D}(\overline{K}(\mathrm{Hecke}(G)))$ is enriched over FinVect.

The Fundamental Theorem of Hecke Operators reveals this FinVect-enriched category to be something very familiar:

Definition 17. *Let* Perm(G) *be the category where:*

- *Objects are* **permutation representations** *of G: that is, representations of G arising from actions of G on finite sets via the 'free vector space' functor* $F\colon \mathrm{FinSet} \to \mathrm{FinVect}$*.*
- *Morphisms are* **intertwining operators** *— that is, G-equivariant linear operators.*

Theorem 18. *For any finite group G,*

$$\overline{D}(\overline{K}(\mathrm{Hecke}(G))) \simeq \mathrm{Perm}(G)$$

The point of this theorem is that the familiar category Perm(G) is obtained from the Hecke bicategory Hecke(G) by a 'degroupoidification' process, namely $\overline{D}$, after a slight change in viewpoint, namely $\overline{K}$.

Acknowledgements

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References

[1] J. Baez and J. Dolan, From finite sets to Feynman diagrams, in *Mathematics Unlimited – 2001 and Beyond*, vol. 1, eds. Björn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29-50. Also available as `arXiv:math.QA/0004133`.

[2] J. Baez and J. Dolan, the Geometric Representation Theory Seminar, Fall 2007.