"Just in time delivery – theorems on the fly"

Theorem – Let G be a group & X, Y finite G-sets. Let $\hom(X,Y)$ be the set of isomorphism classes of spans of G-sets. If G is finite, this is a finitely generated free $\mathbb{N}$-module & we have a map

$$\hom(X,Y) \longrightarrow \{\text{intertwining operators } f: K^X \to K^Y\}$$

for any field K, & the resulting map

$$\hom(X,Y) \otimes_{\mathbb{N}} K \longrightarrow \{\text{intertwining operators } f: K^X \to K^Y\}$$

is onto. (Not necessarily 1-1.)

To state a better theorem like this, it's good to introduce "spans of groupoids".

Last time we considered Pascal's triangle:

$$\begin{tikzcd}
 & \binom{n}{k}_F \arrow[dl, "Y"'] \arrow[dr, "X"] & \\
\binom{n+1}{k}_F & & \binom{n+1}{k+1}_F
\end{tikzcd}$$

Here X & Y are spans of finite sets:

$$Y: \quad \begin{tikzcd}
 & \binom{n}{k}_F \arrow[dl, "1"'] \arrow[dr] & \\
\binom{n}{k}_F & & \binom{n+1}{k}_F
\end{tikzcd} \qquad V \subseteq F^n \longmapsto V \subseteq F^n \oplus F$$

It's awkward to treat it as a span of G-sets since $GL(n, F)$ acts on $\binom{n}{k}_F$ but $GL(n+1, F)$ acts on $\binom{n+1}{k}_F$. We have a homomorphism

$$\varphi : GL(n,F) \longrightarrow GL(n+1,F)$$

A <u>groupoid</u> is a category where all morphisms are invertible. Any category C has an underlying groupoid C_0, with the same objects but only isomorphisms.

E.g.

$$\mathrm{FinSet}_0 \simeq \coprod_{n\in\mathbb{N}} n!$$

Def. – A <u>group</u> is a 1-object groupoid.

$$(\mathrm{FinVect}_F)_0 \simeq \coprod_{n\in\mathbb{N}} GL(n,F)$$

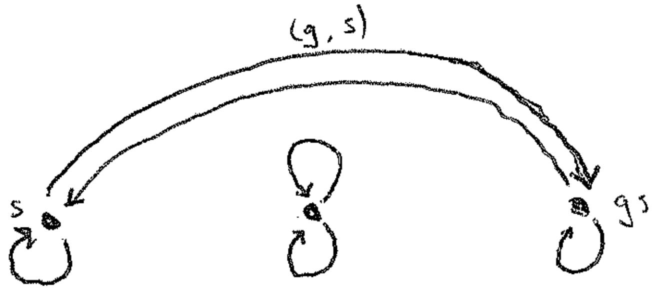
We can switch from thinking about spans of G-sets to thinking about spans of groupoids.

$$G\text{-set } S \longmapsto \text{groupoid } S/\!/G$$

G-set S
G'-set S'
homo $\varphi : G \to G'$
& map $\phi : S \to S'$
s.t. $\phi(gs) = \varphi(g)\phi(s)$

$\longmapsto$ functor $\Phi : S/\!/G \to S'/\!/G'$

Given a set S on which a group G acts, how to build a groupoid $S/\!/G$.



$G = \mathbb{Z}_2$

$S = 3$

The objects of $S/\!/G$ are the elements of S; the morphisms are pairs $(g, s) \in G \times S$ with:

$$(g, s) : s \longrightarrow gs$$

$$\begin{tikzcd} & gs \arrow[dr, "{(h, gs)}"] & \\ s \arrow[ur, "{(g, s)}"] \arrow[rr, "{(hg, s)}"'] & & hgs \end{tikzcd}$$

(composition)

$$(g, s)^{-1} = (g^{-1}, gs)$$

(inverses)

Example: $GL(n, F)$ acts on $\binom{n}{k}_F$ so we get a groupoid:

$$\binom{n}{k}_F /\!/ GL(n, F)$$

Objects are: k-dim $V \subseteq F^n$

A morphism from $V \subseteq F^n$ to $V' \subseteq F^n$ is an element $g \in GL(n, F)$ with $gV = V'$.

In fact, this groupoid is equivalent to the groupoid where the objects are n-dimensional vector spaces (over F) equipped with k-dim subspaces & a morphism

$$f : (\underset{k\text{-dim}}{V} \subseteq \underset{n\text{-dim}}{W}) \longrightarrow (\underset{k\text{-dim}}{V'} \subseteq \underset{n\text{-dim}}{W'})$$

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d] & W \arrow[d, "\wr"', "f"] \\
V' \arrow[r, hook] & W'
\end{tikzcd}$$

i.e. iso $f : W \to W'$ s.t. $f(V) = V'$.

This is the groupoid of "n-dim vector spaces equipped w: k-dim subspace."

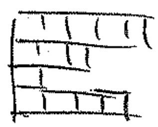
More generally:

$$\binom{n}{n_1, \dots, n_k}_F /\!\!/ GL(n, F) \simeq$$

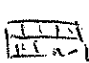
the groupoid of vector spaces over F equipped with D-flag

$$\wr$$

$$D(F^n) = \left\{ \begin{array}{l} D\text{-flags} \\ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = F^n \\ \dim(V_i / V_{i-1}) = n_i \end{array} \right\} \simeq D(\mathrm{Vect}_F)$$

$D =$  $\longleftarrow$ n_i boxes in the i^{th} row

$$\begin{tikzcd}
\binom{n}{k} \in \mathbb{N} \arrow[r, mapsto] \arrow[d, mapsto] & \binom{n}{k}_q \in \mathbb{N}[q] \arrow[d, mapsto] \\
\binom{n}{k} \in \mathrm{FinSet} \arrow[r, mapsto] \arrow[d, mapsto] & \binom{n}{k}_{F_q} \in \mathrm{FinSet} \arrow[d, mapsto] \\
\binom{n}{k}_{\mathrm{Set}} \in \mathrm{Gpd} \arrow[r, mapsto] & \binom{n}{k}_{\mathrm{Vect}_{F_q}} \in \mathrm{Gpd}
\end{tikzcd}$$

D-flagged finite sets where $D =$  $\begin{array}{l} k \\ n-k \end{array}$

$\|$

$D(\mathrm{Set})$ — $\binom{n}{k}_{\mathrm{Set}}$

$\binom{n}{k}_{\mathrm{Vect}_{F_q}}$ — $D(\mathrm{Vect}_{F_q})$

Next: how

$$\phi : S \longrightarrow S' \qquad \varphi : G \longrightarrow G' \quad \text{s.t.} \quad \ldots$$

gives a functor

$$\Phi : S/\!/G \longrightarrow S'/\!/G'$$

& thus a span of G-sets gives a span of groupoids.