"Hecke Operator"

We want to use Hecke operators to find irreducible representations inside of bigger representations.

Thm:

G finite group, R doubly transitive permutation rep of G

Then R is the direct sum of 2 irreps, one of which is the 1-dim trivial rep.

\[ G \rightarrow S! \text{ induces } G \rightarrow [S^n]! , \]

which has only two orbits, one of which is the diagonal.

"Schur's Lemma" + "Maschke's Theorem"

"The irreps of a finite group G form an orthonormal basis for the 2-Hilbert space of fin-dim reps of G".

We need an "inner product" for this 2-Hilbert space \( \text{Hom}(R, R') \)

which gives a vector space, which is like a
categorified number.

Shur's lemma then says that if you Hom an irrep into an isomorphic irrep, you get a 1-dim rep. If you Hom an irrep into a different irrep, you get 0.

\[ \text{Hom}(R, R) = k^2 \text{ a 2-dim vector space,} \]

since \( R \) is doubly transitive.

\[ 1^2 + 1^2 = 2 \]

Another example:

"Find all the irreps of 4!"

We are going to use

"Gram-Schmidt Orthogonalization"

Consider all combed 4-box Young diagrams

\[
\begin{array}{cccc}
2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

24 12 6 4
$$\text{Hom}(V, W)$$

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HW: Calculate 7 relations in $F, F$. 