Lie Theory Through Examples

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Lecture 5

1 The Spin Groups

The rotation group SO(3) is not simply connected: we have

$$\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}.$$

We can see this most vividly using the famous 'coffee cup trick'. Grab a coffee cup, turn it around a full 360 degrees, and your arm has a twist in it. But, there's a way to turn it around *another* full 360 degrees and have your arm back to normal. Your arm describes a path in the rotation group, going from the identity at your shoulder to the rotation of the cup at your hand. So, you can see that rotating the cup 360 degrees once gives a noncontractible loop in SO(3), but doing it twice gives a contractible loop: your arm automatically does the contraction.

By general facts about Lie groups, this means there's a simply-connected Lie group with a 2-1 homomorphism onto SO(3). In fact this group is just SU(2), as explained earlier.

This idea generalizes: for any $n \geq 3$, we have

$$\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}.$$

so there is a simply-connected group with a 2-1 homomorphism onto $\mathrm{SO}(n)$. This is called the **spin group** $\mathrm{Spin}(n)$. It's a simply-connected compact simple Lie group, so all the general ideas we've been discussing apply.

What are the spin groups like? To really get our hands on them, it's good to use Clifford algebras. I love Clifford algebras, so in an ideal version of this course I would spend a month or two explaining Clifford algebras before discussing the lattices associated to the spin groups. But, we can get surprisingly far just knowing that each rotation group $\mathrm{SO}(n)$ has a simply-connected double cover! So, that's what we'll do.

Let's start by working out the lattice associated to Spin(3). We've already done this, since $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, but let's do it a different way, that will work for other spin groups.

Let's find a maximal torus in Spin(3). We'l use the fact that since Spin(3) has a 2-1 homomorphism onto SO(3):

$$p\colon \mathrm{Spin}(3) \to \mathrm{SO}(3)$$

any maximal torus in Spin(3) maps down to a maximal torus in SO(3), and the inverse image of any maximal torus in SO(3) is a maximal torus in Spin(3).

Can we find a maximal torus in SO(3)? This is a maximal bunch of rotations that commute with each other. We can take all the rotations around any axis, say the z axis:

$$T = \{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R}\}.$$

Note that

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, the Lie algebra of the maximal torus is

$$\mathfrak{t} = \{ \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \theta \in \mathbb{R}\}$$

and the kernel of the map

$$\begin{array}{cccl} e\colon & \mathfrak{t} & \to & T \subseteq \mathrm{SO}(3) \\ & x & \to & \exp(2\pi x) \end{array}$$

is:

$$\{ \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{Z}\}.$$

This calculation is for SO(3), but we need to do the corresponding calcualtion for Spin(3). Since Spin(3) is a covering group of SO(3), it looks locally just like SO(3), so its Lie algebra is the same — or more precisely, isomorphic. So,

$$\mathfrak{spin}(3) \cong \mathfrak{so}(3)$$

so we'll just identify $\mathfrak{spin}(3)$ with $\mathfrak{so}(3)$. Similarly, we get the same $\mathfrak{t}$. But, we get a different lattice! Note that for any $a \in \mathbb{Z}$,

$$e(\begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponds to rotating a full turns around the z axis. This is always the identity in SO(3), but it's only the identity in Spin(3) when a is *even!* That's because Spin(3) is a a double cover of SO(3).

So, if we let L be the kernel of

$$\begin{array}{cccl} e\colon & \mathfrak{t} & \to & T \subseteq \mathrm{Spin}(3) \\ & x & \to & \exp(2\pi x) \end{array}$$

we see that

$$L = \{ \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in 2\mathbb{Z}\}.$$