Category $\mathcal{C}$

functor $f: \mathcal{C} \to \mathcal{C}'$

natural transformation.

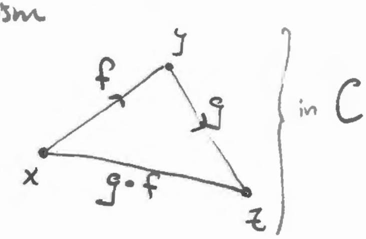
$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$$

Duality:

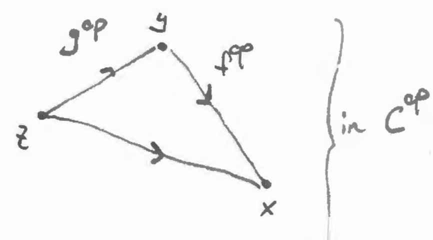
Every category C has an opposite, C^{op}. (morphisms

C^{op} has the same objects but the morphisms are "Turned around".

There is a 1-1 correspondance between morphisms in C and morphisms in C^{op}, with $f: X \to Y$ in C corresponding to morphism $f^{op}: Y \to X$ in C^{op}



We compose morphisms in C^{op} by: $f^{op} \circ g^{op} := (g \circ f)^{op}$



Strangely enough, C, and C^{op} can look very different.

The study of how categories C relate to their partners C^{op} is called duality.

(Its like dual spaces for finite dimensional vector spaces)

It turns out that the dual of geometry is algebra!

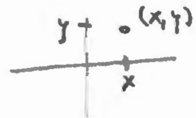
why?

Geometry: Study collections of points and structures on them.

Algebra: Study Addition & Multiplication.

Descartes realized geometry can be turned/Reduced to algebra (Analytic Geometry)

We can associate to any finite dimensional vector space V (over $\mathbb{R}$)
a commutative ring $\mathcal{O}(V)$ consisting of all polynomial functions on V
w. usual $+$ & $\cdot$



← Des(cartesian) plane

So Geometry (space) $\Rightarrow$ Algebra

$V = \mathbb{R}^n \Rightarrow \mathcal{O}(V)$ consists of polynomials in the coordinate functions $x_1, \dots, x_n$

$$\mathcal{O}(V) = \mathbb{R}[x_1, \dots, x_n]$$

Then we can describe certain subspaces of V

as $X \xrightarrow[1\text{-}1]{} V$

or as Quotient algebras of $\mathcal{O}(V)$

as $\mathcal{O}(V) \xrightarrow[\text{onto}]{} \mathcal{O}(X) = \mathcal{O}(V)/I$ ← an ideal

<u>example</u> $S^1 \to \mathbb{R}^2$ unit circle is a subspace of the plane.

$$S^1 = \{(x,y) : x^2+y^2-1 = 0\}$$

But we also have an algebra $\mathcal{O}(S^1)$ of polynomial functions on the unit circle,
with $\mathcal{O}(S^1) = \mathbb{R}[x,y]/\langle x^2+y^2-1\rangle$ when $\langle x^2+y^2-1\rangle$ is the ideal generated by x^2+y^2-1

So the map $S^1 \xrightarrow{1\text{-}1} \mathbb{R}$ gets flipped to $\mathcal{O}(\mathbb{R}^2) \to \mathcal{O}(S^1)$ $\left\{ \begin{array}{l} \text{which is just restriction} \\ f \in \mathcal{O}(\mathbb{R}^2) \\ \text{gives } f|_{S^1} \in \mathcal{O}(S^1) \end{array} \right.$

Moreover $f, g \in \mathcal{O}(\mathbb{R}^2)$ restrict to the same function on S^1 iff
$f-g \in \langle x^2+y^2-1\rangle$ i.e. $f-g = (x^2+y^2-1)\,h(x,y)$

This begins a huge bunch of Ideas!

Algebraic Geometry: the study of geometry using commutative ring.

Our idea: Subspaces of V should correspond to quotient rings of $\mathcal{O}(V)$ or ideals of $\mathcal{O}(V)$

Some problems.

example: 1 $\langle x^2+y^2+1\rangle \subseteq \mathcal{O}(\mathbb{R}^2)$

Two diff. Ideals correspond to the same subspace

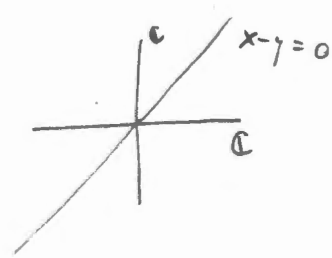
clearly $x^2+y^2+1 \neq 0$ for any point in $\mathbb{R}^2$, but it seems that this description of the $\emptyset$ subspace. is a bit complicated.

we'd like to see a 1-1 correspondence between subspaces of $\mathbb{R}^2$ and ideals of $\mathcal{O}(\mathbb{R}^2)$

Now we have $\langle 1\rangle$ corresponding to $\emptyset$ in $\mathbb{R}^2$ since $\emptyset = \{(x,y) : 1=0\}$

How can we make this work? well, you can use $\mathbb{C}$ instead of $\mathbb{R}$.
But it seems silly to be forced into it.

example: there is a line in $\mathbb{C}^2$ given by $x=y$ with ideal $\langle x-y\rangle \subseteq \mathbb{C}[x,y]$



But $(x-y)^2$ also vanishes only when $x-y=0$

So we get a different ideal that defines the same subspace.
$\langle (x-y)^2\rangle \subseteq \mathbb{C}[x,y]$

So using $\mathbb{C}$ doesn't really help unless you use the radical of an ideal.

but Grothendieck found a solution that was better.

Instead, he cut the Gordian Knot and defined a new type of Space called an affine scheme such that the correspondance between algebra and geometry is perfect.

We'll make a category AffSch where objects are "affine Schemes" and morphisms are maps between them, S.T.

$$(\text{AffSch})^{op} = \text{CommRing}$$

let $\text{AffSch} : (\text{CommRing})^{op}$

<u>example</u> the circle is an affine scheme, namely the Comm ring:

$$\mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle$$

<u>example</u> plane is $\mathbb{Z}[x,y]$ and drop $\mathbb{R}$.

now the "circle is included in the plane" means that we have a homomorphism of commutative rings: $\mathbb{Z}[x,y] \to \mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle$

namely the quotient map.

we also have $\mathbb{R}[x,y] \to \mathbb{R}[x,y]/\langle x^2+y^2-1\rangle$

In noncommutative geometry, we try to invent some new kind of "space" so that $\text{AffSch} = \text{CommRing}^{op}$ gets generalized to something like $\_\_\_ = \text{Ring}^{op}$

<u>Example</u> in classical mechanics, a particle on the line has position q, momentum p.

So the <u>state</u> of a particle is a point (q,p) in the plane.

The nicest functions on the plane are called "observables" for the particle : maybe energy.

the simplest functions are polynomials $\mathbb{R}[q,p]$

If we make $pq = qp + iK$ then were doing quantum mechanics.

[Denotes category of]

Geometry	Commutative Algebra
Algebraic geometry $C = [\text{Affine Schemes}]$	Ring Theory $C^{op} = [\text{commutative rings}]$
Topology $C = [\text{compact Hausdorff Spaces}]$	C^*-algebra Theory: $C^{op} = [\text{commutative } C^*\text{-algebras}]$
Set Theory $C = [\text{Sets}]$	Logic $C^{op} = [\text{atomic boolean algebras}]$

example

$$\mathrm{CHaus} = [\underset{\text{object}}{\text{compact Hausdorff Space}}, \underset{\text{morphisms}}{\text{Continuous maps}}]$$

From a compact Hausdorff space X, an algebra

$C(X) = \{ f : X \to \mathbb{C} . f \text{ continuous} \}$ is an algebra with $(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x) g(x)$

$(cf)(x) = c \cdot f(x) \quad c \in \mathbb{C}$

its commutative! It's also a $*$-algebra with $f^*(x) = \overline{f(x)}$

a) $(f+g)^* = f^* + g^*$

$(fg)^* = g^* f^*$

$(cf)^* = \bar{c} f^*$

Also $C(X)$ has a norm: $\|f\| = \sup_{x \in X} |f(x)|$ makes sense since X is compact.

This makes $C(X)$ into a C^*-algebra, meaning:

$$\|f g\| \leq \|f\| \cdot \|g\|$$

$$\|f^*\| = \|f\|$$

$$\|f^* f\| = \|f\|^2 \quad (C^*\text{-Axiom})$$

Since $\|f^* f\|$

$$= \sup (f^* f)(x) = \sup |f(x)|^2$$

$$= \left(\sup |f(x)|\right)^2$$

$$= \|f\|^2$$

so $C(X)$ is a commutative C^*-algebra.

(i.e. a continuous map)

Next, can we take a morphism between compact Hausdorff spaces $f : X \to Y$

and Turn it into homomorphism of commutative C^*-algebras.

A homomorphism between C^*-algebras, say $F : A \to B$, is a map s.t.

1. $F(a+b) = F(a) + F(b)$
2. $F(cb) = F(c)\,F(b)$
3. $F(ca) = c\,F(a)$
4. $F(a^*) = F(a)^*$
5. $\exists\, k > 0$ s.t. $\|F(a)\| \leq k\,\|a\|$ $\forall a$

$\Downarrow$

$\|F(a)\| = \|a\|$

{Think bounded maps & Banach spaces — $F \in X^*$}

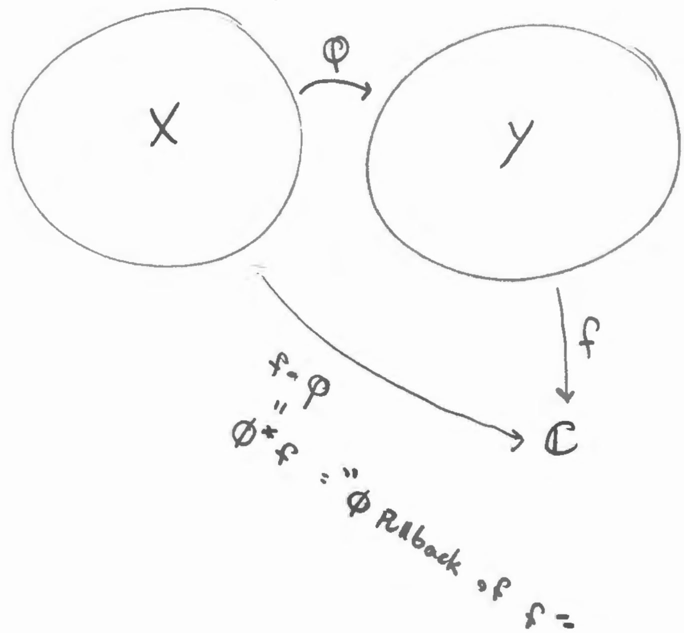
So we get a Category $\mathrm{Comm}\,C^*\mathrm{alg} = [\mathrm{Comm}\,C^*\mathrm{alg.},\ C^*\text{-algebra homomorphisms}]$

But How does a cont. map $\varphi : X \to X$ between compact Hausdorff spaces

give a C^*-alg Homomorphism between $C(X)$ & $C(Y)$

We'll get one $\varphi^*: C(Y) \to C(X)$ by: $\varphi^*(f)(x) = f(\varphi(x))$: $f \in C(Y)$, $x \in C(X)$

or: $\varphi^*(f) = f \circ \varphi$



This is why algebra is the dual of geometry. It goes backwards!

$\phi: X \to Y$

$\phi^*: C(Y) \to C(X)$

Also $(\phi\psi)^* = \psi^* \phi^*$

So we are getting a functor:

$$C: \mathrm{CHaus} \longrightarrow (\mathrm{Comm}\ C^* \mathrm{Alg})^{op}$$

$$X \longrightarrow C(X)$$

$$\phi: X \to Y \longmapsto \varphi^*: C(Y) \to C(X)$$

$\boxed{\text{Thm}}$ [Gelfand - Naimark]

This functor is an equivalence of categories

There is a functor going back:

$$\mathrm{Spec}: \mathrm{Comm}\ C^* \mathrm{Alg}^{op} \longmapsto \mathrm{CHaus}.$$

{ Spec for Spectrum.

s.t. $\mathrm{Spec} \circ C \cong 1_{\mathrm{CHaus}}$

↳ Natural Isomorphism.

$C \circ \mathrm{Spec} \cong 1_{\mathrm{comm}C^*\mathrm{alg}}$

What is Spec?

given a commutative C^* algebra A how do we get a space $\mathrm{Spec}(A)$

lets Try $A = C(X)$ then $\mathrm{Spec}(C(X))$ should be X

How do we recover the points of X from $C(X)$

what's a point of X? $\left\{ \begin{array}{l} \text{should either be a space} \\ \text{or a map} \end{array} \right\}$

$\varphi : \{*\} \longrightarrow X$ where $\{*\}$ is the one point space

i.e. given $x \in X$ there is a map $\varphi : \{*\} \to X$

$* \mapsto x$

So for each point we get a map & conversely any map determines a point.

Our functor $C : \mathrm{CHaus} \to \mathrm{Comm}\, C^*\mathrm{alg}$

will Turn $\varphi : \{*\} \to X$ into a Homomorphism

$$\varphi^* : C(X) \to C(\{*\})$$

$$f \longmapsto f \circ \varphi$$

In fact $C\{*\} \cong \mathbb{C}$ where $g \in C\{*\}$ gives $g(*) \in \mathbb{C}$

So we get $\varphi^* : C(X) \to C(\{*\}) \cong \mathbb{C}$

$$f \to f \circ \varphi \mapsto \underset{\substack{\| \\ f(x)}}{f \circ \varphi(x)}$$

complicated

simple!

Starting from a point $x \in X$ gives a homomorphism $C(X) \to \mathbb{C}$

$f \longmapsto f(x)$

so a point gives a homomorphism from $C(X)$ to $\mathbb{C}$

Given a commutative C^*-algebra how can

$\boxed{\text{Lemma}}$ 2 distinct points of X give distinct homomorphisms $C(X) \to \mathbb{C}$

(i.e. enough continuous functions to seperate points, notice we couldn't do this if X was not Hausdorff)

$\boxed{\text{Lemma}}$ Any C^*-alg homomorphism from $C(X) \to \mathbb{C}$ comes from a point. by $\psi(f) = f(x)$

(Existence & uniqueness, ... and one)

we get a 1-1 correspondence between points $x \in X$ & homoms of

So given any comm C^* alg. we define a set of points

$$\text{Spec}(A) = \{\psi : A \to \mathbb{C} : \psi \text{ is a } C^* \text{ alg. homom}\}$$

The Topology that makes $\text{Spec}(A)$ into a compact Hausdorff spaces.

ψ_a converges to ψ iff $\psi_i(a)$ converges to $\psi(a)$ $\forall a \in A$

notice $\psi_i(a)$ is in $\mathbb{C}$

Finally, given a C^* alg. homom. $F: A \to B$, How do we get a map of Spaces $\mathrm{Spec}(F): \mathrm{Spec}(B) \to \mathrm{Spec}(A)$?

$\psi: B \to \mathbb{C}$ C^*-alg hom.

$a \in A$

$$\underbrace{\mathrm{Spec}(F)(\psi)(a)}_{\text{in Spec}(A)} = \underbrace{\psi(\underbrace{F(a)}_{\in B})}_{\in \mathbb{C}}$$

So we get functors

$$\begin{tikzcd} \text{CHaus} \arrow[r, bend left, "C"] & \text{Comm } C^* \text{ alg.} \arrow[l, bend left, "\text{Spec}"] \end{tikzcd}$$

which are inverses up to isomorphism,

Duals we've talked about:

Note:

$$\begin{tikzcd} \text{Spaces} \arrow[r, bend left, "\text{functions on the space}"] & \text{Comm Rings} \arrow[l, bend left, "\text{Spectrum}"] \end{tikzcd}$$

$$\begin{tikzcd} \text{point in a space} \arrow[r, bend left] & \text{Homomorphism from a Commutative ring to a field} \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \text{Subspace (Inclusion)} \arrow[r, bend left] & \text{Surjection or Epimorphism} \arrow[l, bend left] \end{tikzcd}$$ or an ideal

[Duality between set theory & Logic]

Basic Idea:

Subset operations	Logical Operations	
$\cup$	$\vee$ (or)	
$\cap$	$\wedge$ (and)	
ϕ	$F = 0$	(You're never in $\emptyset$)
X	$T = 1$	(You're always in X)
complement	$\neg$ (not)	

e.g.

$x \in S \cup T \iff x \in S \vee x \in T$

let's fit this into the same **mold** as last time. We saw that given a compact Hausdorff set we get a commutative C^* algebra

$C(X) = \hom_{Top}(X; \mathbb{C})$ $\left\{\begin{array}{l}\text{continuous maps from} \\ X \text{ to } \mathbb{C}\end{array}\right\}$ notice $\mathbb{C}$ is a Topological space here.

Also given a comm. C^* algebra A,

$\mathrm{Spec}(A) = \hom_{CommC^*alg}(A, \mathbb{C})$ $\left\{\begin{array}{l}\text{homomorphism of} \\ C^*\text{-algebras}\end{array}\right\}$ $\mathbb{C}$ is a C^*-algebra here

we call $\mathbb{C}$ a dualizing object in this context. "The same object X in 2 different categories $\mathcal{C}$ & $\mathcal{D}$ s.t. $\mathcal{C} \longrightarrow \mathcal{D}^{op}$

$c \longrightarrow \hom_{\mathcal{C}}(c, X) \in \mathcal{D}$

for some strange reason

Conversely $\mathcal{D}^{op} \to \mathcal{C}$

$d \to \hom_{\mathcal{D}}(d, x) \in \mathcal{C}$

& these maps are inverses

so $\mathcal{C}$ & $\mathcal{D}^{op}$ are equivalent.

we'll convert Boolean algebras to sets & vice versa

we just need to figure out the dualizing object. Last time $\mathbb{C}$ worked best as a dualizing object between C^*-algebras and Topological spaces.

we'll ~~??~~ relate sets and Boolean algebras using $2 = \{0,1\} \cong \{F,T\}$

Boolean Algebras are a bit like C^*-algebras, but:

comm C^*-algebras	Boolean algebras
$+$	$\vee$
$\cdot$	$\wedge$
0	F
1	T
$\mathbb{C}$	$2 = \{F,T\}$

Think powerset & {functions representing all subsets of X}

So given a set X, we'll make a Boolean algebra $2^X = \hom_{Set}(X,2)$

An elt. here is a function $f: X \to \{0,1\}$ any subset $S \subseteq X$ gives such a function

$$\chi_S(x) = \begin{cases} 0 & x \notin S \\ 1 & x \in S \end{cases}$$ and conversely any such function gives a subset

The operations $\cup, \cap, {}^c$, on subsets of X correspond to operations $\vee, \wedge, \neg$ on $f: X \to \{0,1\}$

$\chi_{S \cup T} = \chi_S \vee \chi_T$ where $\chi_S \vee \chi_T(x) = \chi_S(x) \vee \chi_T(x)$ and so on.

we can define $\leq$ w. $\wedge, \vee$ etc.

note: $S \subseteq T \iff \chi_S \leq \chi_T$

$\chi_S \leq \chi_T \iff \chi_S \wedge \chi_T = \chi_S$

$\iff \chi_S \vee \chi_T = \chi_T$

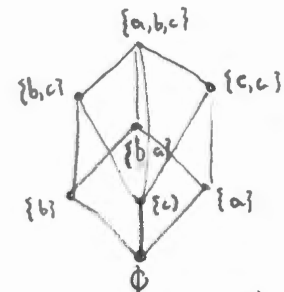
i.e. if $\chi_S(x) = 1$ then $\chi_T(x) = 1$

$(2^S, \vee, \wedge, 0, 1)$ will be a Boolean Algebra

$\boxed{\text{Def}}$ A partially ordered set, A is called a <u>lattice</u> if every pair $a, b \in A$ has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$ also a least element, $0 = F$ and a greatest element $1 = T$

$\boxed{\text{Def}}$ A distributive Lattice is one where $\wedge$ & $\vee$ distribute over each other.

$\boxed{\text{Def}}$ A Boolean Algebra is a distributive lattice A, where every element $x \in A$ has a complement $\neg x$ s.t. $x \wedge \neg x = F$ $\quad x \vee \neg x = T$



(notice if a complement exists it is unique)

<u>example</u> for any set S, 2^S is a Boolean Algebra w. pointwise defined $\leq$: given $f, g \in 2^S$

we say $f \leq g$ iff $f(x) \leq g(x) \;\forall\, x \in S$

It thus has pointwise defined $\vee, \wedge, 0, 1, \neg$ $\quad$ e.g. $(\neg f)(x) = \neg f(x)$

But not every Boolean Algebra is isomorphic to one of this form!

The Boolean Algebras of the form 2^S are called "complete atomic Boolean Algebras"

$\boxed{\text{Def}}$ A complete Boolean Algebra A is one where every $S \subseteq A$ has a least upper bound $\bigvee_{x \in S} x$ & greatest lower bound $\bigwedge_{x \in S} x$ and these distribute over each other

$\boxed{\text{Def}}$ An <u>atom</u> in a Boolean algebra is an element $x \in A$ s.t. $x \neq 0$ and if $y < x$ then $y = 0$

e.g in 2^{S} atoms correspond to singletons $\{s\} \subseteq S$

$\boxed{\text{Def}}$ A Boolean algebra atomic if for each $x \in A$, $x = \bigvee_{\lambda \in \Lambda} y_\lambda$ $y_\lambda \in A$ and y_λ atoms.

There is a category of CABA of complete atomic Boolean algebras & homomorphisms of complete Boolean algebras: $\Phi : A \to B$ preserving $\vee, \wedge, 0, 1, \neg, \bigvee, \bigwedge$

There is a Category Set of Sets and functions

$\boxed{\text{Thm}}$ Set is equivalent to $CABA^{op}$ via these functors

$$\begin{aligned} \text{Set} &\longrightarrow CABA^{op} \\ S &\longmapsto 2^S = \hom_{Set}(S, 2) \end{aligned}$$

$$\begin{aligned} CABA^{op} &\longrightarrow \text{Set} \\ A &\longmapsto \hom_{CABA}(A, 2) \end{aligned}$$

where $2 = \{F, T\}$ is a CABA in the obvious way.

Given function $\varphi : S \longrightarrow T$ we get a complete Boolean algebra hom.

$$\varphi^* : 2^T \to 2^S \qquad \Phi^*(f)(s) = f \circ \varphi(s) \quad f \in 2^T \quad f : T \to 2$$

This is called a pullback

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, bend right, "\varphi^* f = f \circ \varphi"'] & T \arrow[d, "f"] \\ & 2 \end{tikzcd}$$

e.g $\{ f \in L^\infty[0,1] : f(x) = \overset{F}{\overset{\uparrow}{0}} \text{ or } \overset{T}{\overset{\uparrow}{1}} \}$ is complete but not atomic

$\{B \in \mathcal{B}_{\mathbb{R}}\}$ is a Boolean algebra that is not complete since Borel measurable sets is not closed under arbitrary unions or intersections.

Geometry	Algebra
?	Boolean Algebras
?	Linear Algebra (Finite dim. vector spaces)
?	Finite Abelian Groups

The opposite of the category of all Boolean Algebras is the category of Stone spaces: Compact Hausdorff spaces that are totally disconnected (Every open set is closed and vice-versa)

The Boolean Algebra of a Stone Space consists of its open subsets, with

$A \cup B$ as "$\vee$"

$A \cap B$ as "$\wedge$"

A^c as "$\neg$"

Let FinVect be the category of finite dimensional vector spaces over your favorite field and linear maps as morphisms.

what's $(\text{FinVect})^{op}$? A typical morphism in FinVect is $T: \mathbb{R}^n \to \mathbb{R}^m$

So a morphism in $(\text{FinVect})^{op}$ is $T^{op}: \mathbb{R}^m \to \mathbb{R}^n$

which is suspiciously similar to the transpose.

In fact $(\text{FinVect})^{op} \xrightarrow{\sim} \text{FinVect}$

$$T: \mathbb{R}^n \to \mathbb{R}^m \longmapsto T^T: \mathbb{R}^m \to \mathbb{R}^n \qquad \text{i.e. } (T^T)^{op}: \mathbb{R}^n \to \mathbb{R}^m \text{ in } (\text{FinVect})^{op}$$

We can also get the equivalence $\text{FinVect} \simeq (\text{FinVect})^{op}$ using $\mathbb{R} \in \text{FinVect}$ as an dualizing object:

$$\text{FinVect} \dashrightarrow (\text{FinVect})^{op}$$

$$V \longmapsto \text{hom}(V, \mathbb{R}) = V^*$$

$$T: V \to W \longmapsto T^*: W^* \to V^* \text{ in FinVect}$$

$$(T^*)^{op}: \quad \ldots \quad \text{ in } (\text{FinVect})^{op}$$

So Finvect straddles the worlds of Geometry and Algebra, by being its own "op"

well, that makes sense by the way it is taught. First you draw pictures then

prove things by computation and view vectors as collections of numbers.

Also the Category [finite abelian groups and group homomorphisms] is its own op.

[Galois Theory]

Galois Theory is secretly about dualities between posets.

$\boxed{\text{Def}}$ a poset is a partially ordered set $(S, \leq)$ where $\leq$ is reflexive, transitive and antisymmetric.

If $(S, \leq)$ is a poset, we get a category with elts. of S as objects and there exists a unique morphism $f: x \to y$ iff $x \leq y$ $(x, y \in S)$, and no morphisms otherwise.

In fact, the categories we get this way are precisely those with ① at most one morphism from any object x to any object y, and ② if there are morphisms from x to y and y to x then $x = y$

So to a category theorist, a poset is a category with these two properties.

Given categories of this sort, a functor is really just an order preserving map $f: (S, \leq) \to (T, \leq)$ i.e. a function s.t. $\underset{\text{in } T}{f(x) \leq f(y)}$ when $\underset{\text{in } S}{x \leq y}$

Given category of this sort coming from the poset $(S, \leq)$ its opposite comes from the poset $(S, \leq^{op})$ when $x \leq^{op} y$ iff $y \leq x$ So we write $(S, \geq)$ as $(S, \leq)$

what are adjoint functors between categories of this sort?

$\boxed{\text{Def}}$ Given categories C, D we say a functor $L: C \to D$ is the left adjoint of a functor $R: D \to C$, or R is the right adjoint of L, if there's a natural 1-1 correspondence

$$x \in C, \; y \in D \qquad \hom_D(Lx, y) \cong \hom_C(x, Ry)$$

example Let $L: \text{Set} \to \text{Group}$ sends S to the free group on S

and $R: \text{Group} \to \text{Set}$ sends any group G to its underlying set.

Here

$$\hom_{Grp}(LS, G) \cong \hom_{Set}(S, RG)$$

example what are adjoint functors between posets $(S, \leq)$ & $(T, \leq)$?

It's a pair of order preserving functions

$$\begin{tikzcd} (S, \leq) \arrow[r, bend left, "L"] & (T, \leq) \arrow[l, bend left, "R"] \end{tikzcd}$$

S.T. $Lx \leq y \Leftrightarrow x \leq Ry$ which comes from $\hom_D(Lx, y) \cong \hom_C(x, Ry)$

$\boxed{\text{Def}}$ a pair of Adjoint functors between posets is called Galois correspondance

$\boxed{\text{Thm}}$ suppose we have a Galois correspondance $\begin{tikzcd} (S, \leq) \arrow[r, bend left, "L"] & (T, \leq) \arrow[l, bend left, "R"] \end{tikzcd}$. Here we get an order preserving map $RL: (S, \leq) \to (S, \leq)$

write $\bar{x}$ for RLx, then $x \leq \bar{x}$ and $\bar{\bar{x}} = \bar{x}$

{behaves like closure}

we say $\bar{\ }$ is a closure operator on the poset $(S, \leq)$

$\boxed{\text{Thm}}$ (continued) {behaves like interior}

Similarly write y° for LRy then $y^\circ \le y$ $\forall y \in T$

and $(y^\circ)^\circ = y^\circ$

Finally, L & R give a bijection between closed elements of S ($x \in S$ s.t. $\bar{x} = x$)

and open elements of T ($y \in T$ s.t. $y^\circ = y$)

[Galois Theory]

Suppose you have any algebraic gadget - a set with some operations obeying axioms : monoids, groups, rings, fields

Then we can define a "subgadget" of a gadget K ~~that~~ to be a subset $k \subseteq K$ closed under all the operations.

The gadgets F with $k \subseteq F \subseteq K$ form a poset with $\subseteq$ as the partial ordering.

Let's call this poset D.

Galois Theory uses groups to study D.

Any gadget K has a group $Aut(K)$ of automorphisms (1-1 onto functions $g: K \to K$ that preserve the operations)

ex: $g(x+y) = g(x) + g(y)$ $g(0) = 0$ $g(1) = 1$ $g(xy) = g(x)\, g(y)$

we say an element $x \in K$ is fixed by $g \in Aut(K)$ if $gx = x$. We say a subgadget $F \subseteq K$ is fixed by an automorphism if $gx = x$ $\forall x \in F$.

Notice $\{ g \in Aut\,K : g \text{ fixes } F \}$ is always a subgroup of $Aut(K)$

We define a Galois group $G(K|k)$ to be the subgroup of $\mathrm{Aut}(K)$ fixing subgadget $k \subseteq K$

let C be the poset of subgroups of $G(K|k)$, where the partial ordering is $\subseteq$.
The idea is to use C to study D.

We'll do this by constructing a Galois correspondence

$$\begin{tikzcd} C \arrow[r, bend left, "L"] & D^{op} \arrow[l, bend left, "R"] \end{tikzcd}$$

i.e. order preserving maps obeying

$LG \subseteq F \Leftrightarrow G \supseteq RF$

what is R? It maps gadgets F s.t. $k \subseteq F \subseteq K$ to subgroups of Galois group $G(K|k)$

It works as follows: $RF = \{ g \in \mathrm{Aut}(K) \text{ s.t. } g \text{ fixes } F \}$

To show $R: D^{op} \to C$ is order preserving (i.e. a functor) we need $k \subseteq F \subseteq F' \subseteq K$

then $RF \supseteq RF'$ {makes sense since if g fixes F' and $F \subseteq F'$ then g fixes F.}

what is L? It maps subgroups $G \subseteq G(K,k)$ to gadgets between k and K
and works as follows:

$$LG = \{x \in K : G \text{ fixes } x\} := \{x \in K : g(x) = x \quad \forall g \in G\}$$

{is a subgadget of k since automorphisms preserve operations}

To show $L: C \to D^{op}$ is order preserving we need: $G \subseteq G' \subseteq G(K|k)$

to imply $LG \supseteq LG'$ {G' is larger so if you are fixed by G' then obviously you're fixed by G.}

next, why is $\begin{tikzcd} C \arrow[r, bend left, "L"] & D^{op} \arrow[l, bend left, "R"] \end{tikzcd}$ a Galois Connection?

why is $LG \subseteq F \iff G \supseteq RF$

What is; $LG \subseteq F$ means everything fixed by G is in F
elt of K

$G \supseteq RF$ means everything fixing F is in G.

These are two ways to say the same thing.

Now we can relate nice subgadgets between $k \subseteq F \subseteq K$ and nice subgroups $G \subseteq G(K|k)$ using the theorem we saw last time.

Thm Suppose you have $C \underset{R}{\overset{L}{\rightleftarrows}} D^{op}$ a Galois Connection.

using C^{op} instead would get interior operators.

Define. $\bar{c} = RLc \qquad c \in C$

$\bar{\partial} = LR\partial \qquad \partial \in D^{op}$ {but elts of D & D^{op} are the same}

Then these are closure operators: $c \leq \bar{c}$ & $\bar{\bar{c}} = \bar{c}$

$\partial \leq \bar{\partial}$ & $\bar{\bar{\partial}} = \bar{\partial}$ ($\leq$ ordering on D)

(we say $c \in C$ is closed if $c = \bar{c}$ (similar for D))

and L & R give a 1-1 correspondence between closed elements of C and closed elements of D.

In our applications, what is a "closed" subgadget $k \subseteq F \subseteq K$?

It's one where $F = LRF = L\{g \in Aut(G) : g \text{ fixes } F\} = \{x \in K : x \text{ fixed by all } g \text{ that fix } F\}$

So a subgadget F is closed if it contains all $x \in K$ that are fixed by <u>all</u> $g \in G(K|k)$ that fix F

what's a closed subgroup $G \in G(K|k)$

$G = RLG$

$= R\{x \in K : x \text{ is fixed by } G\} = \{g \in G(K|k) : g(x) = x \text{ for all } x \text{ fixed by } G\}$

So a subgroup G is closed if it's the group of all $g \in G(K|k)$ that fix all x fixed by G.

The hard part of Galois theory includes finding more concrete characterizations of the closed subfields $k \subseteq F \subseteq K$ & similar for the closed ~~subfields~~ subgroups.

[Groupoids]

$\boxed{\text{Def}}$ A morphism $f: x \to y$ in a category has an inverse $g: y \to x$ if $fg = 1_y$ & $gf = 1_x$

if f has an inverse, its unique so we write f^{-1}.

A morphism with an inverse is called an isomorphism. If there is an isomorphism $f: x \to y$ we say x & y are isomorphic.

$\boxed{\text{Def}}$ A groupoid is a category where all morphisms are isomorphisms.

ex. Any group G gives a groupoid with one object $\{*\}$ and morphisms $g: * \to *$ corresponding to elements $g \in G$ with composition coming from mult. in G.

Conversely any 1-object groupoid gives a group. So "a group is a 1-object groupoid".

More generally, if C is any category & $x \in C$, the isomorphisms $f: x \to x$ form a group called the automorphism group $\mathrm{Aut}(x)$ under composition.

ex Given any category C, there is a groupoid, the core of C, C_0, whose objects are those of C & whose morphisms are the isomorphisms of C, composed as before

ex FinSet : [finite set, functions]

then FinSet_0 : [finite sets, bijections]

and if n is your favorite n-element set, $\text{Aut}(n) = S_n$ (the symmetry group)

So FinSet_0 unifies the symmetric groups into a single structure.

ex say G is a group acting on a set X. $\alpha : G \times X \to X$, $(g,x) \mapsto gx$

Often, people form the quotient set X/G, when an elt $[x]$ is an equivalence class of elts $x \in X$ where $x \sim y$ iff $y = gx$ $g \in G$

A better thing to do is to form the translation (action) groupoid, often written $X/\!/G$, where objects are elts of X and a morphism from x to y is a pair (g,x) where $g \in G$ and $gx = y$

$$x \xrightarrow{g} y$$

The composite of $x \xrightarrow{(g,x)} y$ and $y \xrightarrow{(h,y)} z$ is $x \xrightarrow{(hg,x)} z$

In X/G, we say x & y are equal if $gx = y$ in $X/\!/G$ we say x & y are isomorphic or more precisely we have a chosen isomorphism $(g,x) : x \to y$.

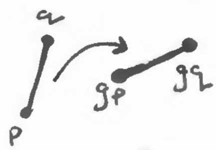
To a first approximation, a moduli space is a set X/G, given some obvious topology, while a moduli stack is a group $X/\!/G$, where the sets of objects and morphisms have topologies.

ex let X be the set of line segments in the Euclidean Plane

Let G be the group of all bijections $g: \mathbb{R}^2 \to \mathbb{R}^2$ S.T. they preserve distances,

so $|gp - gq| = |p - q|$ i.e. g preserves distances.

G acts on X:



more precisely, $X = \mathbb{R}^2 \times \mathbb{R}^2$ G acts on it via $g(p,q) = (gp, gq)$

Notice we're not counting (p,q) as the same as (q,p) and we allow $p = q$

So X/G is the "moduli space of line segments"

$X/G \cong [0, \infty)$ since p — q is isomorphic to q — p i.e. $(p,q) \sim (q,p)$ so they give the same point in X/G

Next, consider $X//G$. Objects are line segments and morphisms are:

$$(p,q) \xrightarrow{(g,p,q)} (p',q')$$

where $g(p) = p'$, $g(q) = q'$

Given a groupoid C, we can form

① the set $\underline{C}$ of isomorphism classes of objects: $[x]$ where $[x] = [y]$ iff $x \cong y$.

② for any $[x] \in \underline{C}$ a group $Aut(x)$, where x is any representative of $[x]$

any two representatives $x \cong y \Rightarrow Aut(x) \cong Aut(y)$ as groups. So it won't matter which x we choose.

$\boxed{\text{Thm}}$ Given a groupoid C, we can recover C (up to equivalence) from $\underline{C}$ and all the groups $Aut(x)$ (one from each isomorphism class in $\underline{C}$)

ex $C = \text{Finset}_0$ $\quad \underline{C} \cong \mathbb{N}$ and for each $n \in \mathbb{N}$ we get the group S_n

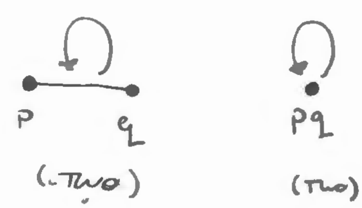
which is isomorphic to $\text{Aut}(x)$ where $x \in \text{Finset}_0$ with n elements.

ex $C = X/\!/G$ where X is the set of line segments & G is the Euclidean group of the plane.

$\underline{C} \cong [0,\infty)$ In general, $\underline{X/\!/G} = X/G$ since both are names for the set of equivalence classes $[x]$ where $x \sim y$ iff $y = gx$, $g \in G$

But $X/\!/G$ has more information, namely all the automorphism groups $\text{Aut}(x)$, one for each equivalence class.

In our example what is $\text{Aut}((p,q))$? an automorphism



$\text{Aut}((p,q))$ is ~~the~~ $\mathbb{Z}_2$ by reflection preserving (p,q)

if $p = q$ you get $O(2)$ of all rotations and reflections of the plane fixing $p \in \mathbb{R}^2$

{Moduli Spaces & Moduli Stacks}

Given a groupoid C, let $\underline{C}$ be the set of isomorphism classes of objects.

Often $\underline{C}$ will have the structure of a space (e.g. a Topological Space, Manifold, Algebraic Variety, ~~or~~ ~~affine~~ Scheme)

$\underline{C}$ is called a moduli space.

ex: if G is a group acting on a set X, we get a groupoid $X/\!/G$, the translation groupoid, where: objects are elements of X and morphisms are pairs (g,x) where $g \in G$, $x \in X$ and $y = gx$.

Then: $\underline{X/\!/G} \cong X/G$, where X/G has elements $[x]$ with $x \sim y$ when $y = gx$ for some $g \in G$

recall we [illegible] a theorem

$\boxed{\text{Thm}}$ The groupoid $X/\!\!/G$ is equivalent to the groupoid with:

- One object $[x]$ for each $[x] \in X/G$
- One morphism $f: [x] \to [x]$ for each morphism $f: x \to x$

where x is any chosen representative of the equivalence class $[x]$

notice if $[x] \neq [y]$ there are no morphisms between them.

we call X/G a moduli space, and $X/\!\!/G$ the moduli stack.

Last time we looked at an example of "The moduli Stack of line segments" in Euclidean Geometry. Here $X = \mathbb{R}^2 \times \mathbb{R}^2 \ni (p,q)$ and $G = O(2) \ltimes \mathbb{R}^2$

Here G is the Euclidean group of the plane and we think of (p,q) as a directed line segment. and we allow $p = q$

Then the moduli space of intervals is $X/G \cong [0, \infty)$, the space of lengths.

$$[(p,q)] \mapsto |p-q|$$

The moduli stack $X/\!\!/G$ keeps track of symmetries of intervals.

$\mathrm{Aut}[(p,q)] \cong \mathrm{Aut}((p,q))$ is the subgroup of G, consisting of all $g \in G$ with $(gp, gq) = (p,q)$

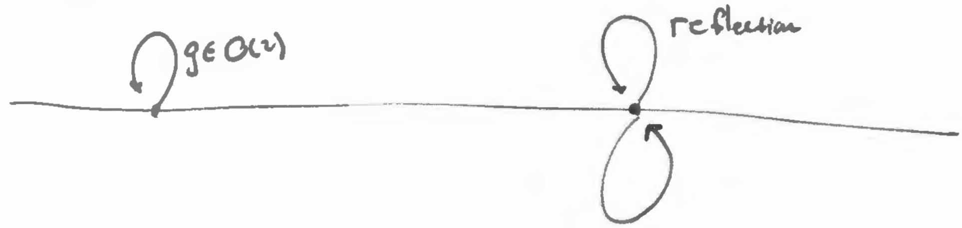
$\mathrm{Aut}((p,q)) \cong \mathbb{Z}_2 \qquad p \neq q$

$\mathrm{Aut}((p,q)) \cong O(2) \qquad p = q$

$\cong \underset{\text{rotations}}{SO(2)} \ltimes \mathbb{Z}_2$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

So the moduli stack looks like:



<u>ex</u> "moduli space of triangles"

use G as the Euclidean group as before, but let X be the set of triangles,

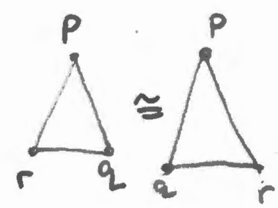
$X = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ where triangles have named vertices that can be equal.

The moduli space X/G is the set of isomorphism classes of triangles

$$X/G \cong [0,\infty)^3$$

$[(p,q,r)] \longmapsto (|p-q|, |q-r|, |r-p|)$

It seems like if (p,q,r) has as automorphisms only the identity.



If we defined a triangle to be an unordered triple of points in $\mathbb{R}^2$, then an equilateral triangles would have S_3 as an automorphism group and isosceles would have $S_2 = \mathbb{Z}/2$

This gives a more interesting moduli stack.

<u>ex</u> A Riemann surface is a 2-dim manifold with charts

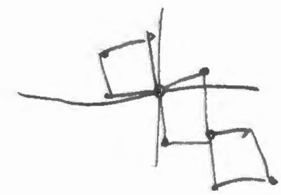
$\varphi_i : U_i \to \mathbb{C}$ s.t. $\varphi_j \varphi_i^{-1}$ is analytic.

every Riemann Surface that is homeomorphic to the plane is isomorphic to $\mathbb{C}$

every Riemann Surface that is homeomorphic to the sphere, is isomorphic to the Riemann Sphere $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$

There are lots of nonisomorphic ways to make a torus into a Riemann Surface. These are called elliptic curves.

Every elliptic curve is isomorphic to one of this form: take a lattice $L \subseteq \mathbb{C}$



i.e. a subgroup of $(\mathbb{C}, +, 0)$ that's isomorphic $\mathbb{Z}^2$, and form $\mathbb{C}/L$, getting a Torus with charts $\varphi_i : U_i \to \mathbb{C}$ and thus an elliptic curve

When do two lattices L, L' give isomorphic elliptic curves.

Answer: when $L' = \alpha L$ for some nonzero $\alpha \in \mathbb{C}$

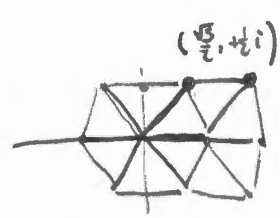
There is a groupoid w. elliptic curves as objects, isomorphisms of Riemann Surfaces as morphisms and we're seeing $\underline{C} \simeq X/\!\!/G$, X is the set of lattices and $G = \mathbb{C}^*$ (nonzero complex #'s w. mult. as group op.)

So X/G is called the moduli space of elliptic curves, and $X/\!\!/G$ is the moduli stack of elliptic curves.

There are two elliptic curves w. larger automorphism than average.

Typical elliptic curve has $\mathbb{Z}/2$  only 180 symmetric

has $\mathbb{Z}/4$ 

has $\mathbb{Z}/6$ 

[Klein Geometry]

$\boxed{\text{Def}}$ A homogeneous G-space for some group G is a G-set X, i.e. a set X w. a map

$$\begin{array}{l} G \times X \to X \\ (g, x) \mapsto x \end{array} \quad \text{s.t.} \quad g_1(g_2(x)) = (g_1 g_2)(x) \qquad 1 \cdot x = x$$

which is transitive i.e. $\forall x, y \in X$, $\exists g \in G$ s.t. $gx = y$

ex in Euclidean geometry $G = O(2) \ltimes \mathbb{R}^2$ is the Euclidean group and $X = \mathbb{R}^2$ is the Euclidean plane with $g = (r, t) \in O(2) \ltimes \mathbb{R}^2$ acting on $x \in \mathbb{R}^2$ by
$gx = rx + t$

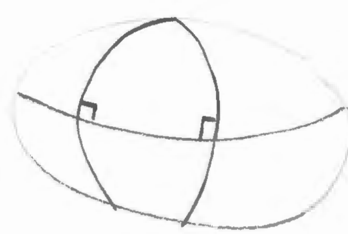
In non-Euclidean geometry, the parallel postulate fails.

Here the Euclidean group is replaced by another group (A 3 dim. Lie group)

Ex in Spherical geometry $G = O(3) = \{g : \mathbb{R}^3 \to \mathbb{R}^3 : g \text{ is linear and } gx \cdot gy = g(x \cdot y) \ \forall x, y \in \mathbb{R}^3\}$

$X = S^2 = \{x \in \mathbb{R}^3 ; x \cdot x = 1\}$

Here we can define a set of lines, namely great circles but any 2 distinct lines intersect in 2 points, so the parallel postulate fails, but other axioms of Euclidean geometry hold.

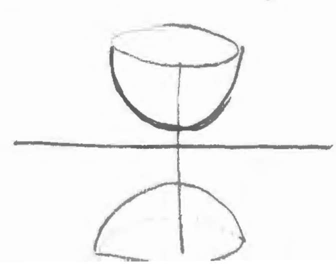


Ex In hyperbolic geometry we let $\mathbb{R}^{2,1}$ be $\mathbb{R}^3$ with the dot product

$$(x, y, z) \cdot (x', y', z') = xx' + yy' - zz'$$

and let $G = O(2,1) = \left\{ g : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1} \ ; \ g \text{ is linear and } gx \cdot gy = g(x \cdot y) \right\}$

and $X = H^2$
$= \left\{ x \in \mathbb{R}^{2,1} : x \cdot x = -1 \right\}$
$= \left\{ (x, y, z) : x^2 + y^2 - z^2 = 1 \right\}$



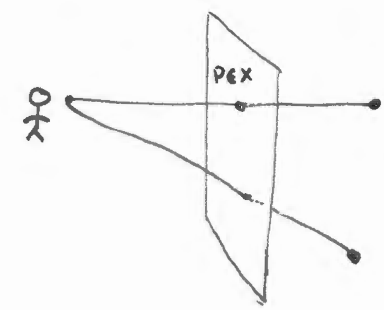
As in spherical geometry, we can define a line to be an intersection of X with some plane through the origin.

The parallel postulate fails because for any line ℓ and p not on ℓ $\exists$ infinitely many lines ℓ' containing p and not intersecting ℓ.

In projective plane geometry, every pair of distinct lines intersects in exactly one point.

In projective geometry, $X = \mathbb{R}P^2$

$= \{\text{lines through origin in } \mathbb{R}^3\}$



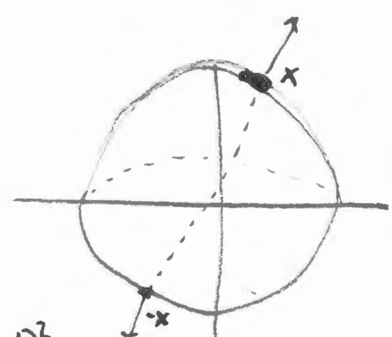
Let $G = GL(3,\mathbb{R}) = \{g : \mathbb{R}^3 \to \mathbb{R}^3 : g \text{ linear, invertible}\}$

or since dilations/transformations $x \longmapsto \alpha x$ $(0 \neq \alpha \in \mathbb{R})$

act as the identity on X. So we can use the projective linear group

$$G = PGL(3,\mathbb{R}) = \frac{GL(3,\mathbb{R})}{\{\alpha I : \alpha \in \mathbb{R}\}}$$

which is an 8 dimensional group.



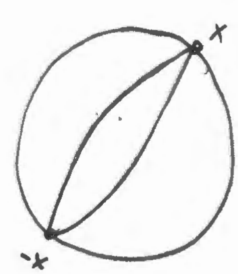
$\mathbb{R}P^2$ can be identified with $S^2/\sim$ where $x \sim y$

iff $y = \pm x$ or therefore with $D^2/\sim$

with $x \sim y$ iff x & y are on the boundary of the disc, D^2

and they are diametrically opposite

or



So $\mathbb{R}P^2$ can be seen as $\mathbb{R}^2$ (homeomorphic to the interior of D^2)

together with "points at infinity" coming from the boundary

of the disc.

We can define a line in $\mathbb{R}P^2$ to be a plane through

the origin in $\mathbb{R}^3$ which contains lots of points in $\mathbb{R}P^2$ (which are lines through

the origin)

Any pair of distinct lines intersect in a unique point and any pair of distinct points lie on a unique line.

Indeed, in projective plane geometry, any theorem has a "dual" version where the role of points and lines are switched.

Note: this is a special case of duality for posets w. $p \le l$ meaning p lies on l

Klein noticed that in all the kinds of geometry we have mentioned so far, we have two homogeneous G-Spaces: the set X of points and also the set Y of lines.

we also get interested in other homogeneous G-Spaces

e.g in 3D geometry we'd have the set of planes, or in 2D ~~projective geometry~~ Euclidean geometry we have the set of flags  i.e. point line pairs where the point lies on the line etc.

So Kliens idea was: a geometry is simply a group, and a type of figure (Point, line, flag) is a homogeneous G-Space X, with an element $x \in X$ being a figure of that type.

we can keep track of all the homogeneous G-spaces by:

$\boxed{\text{Thm}}$ Say X is any homogeneous G-space. Chose an element $x \in X$ and let $H \subseteq G$ be the stabilizer of x i.e. the subgroup $H = \{g \in G : gx = x\}$

then G/H be the set of equivalence classes $[g]$ when $g \sim g' \Leftrightarrow g' = gh$

then G/H is a homogeneous G-space with $g[g'] = [gg']$

and as G-spaces we have $X \cong G/H$

by $\alpha : gx \mapsto [g] \quad \forall g \in G$

with the obvious inverse. Here α is a map of G-Spaces: $\alpha(gx) = g\alpha(x)$.

This allowed Klein to redefine a type of figure to be simply a subgroup of G, since a subgroup $H \leq G$ gives a transitive G-space G/H, and every transitive G-space is isomorphic to one of these.

We've seen that:

- A geometry is a group G
- A type of figure in this geometry is a subgroup $H \leq G$
- The set of figures of that type is $G/_H$: a homogeneous G-space.

How can we do geometry this way? We need G-invariant relations between figures.

ex. projective plane geometry

$G = PGL(3, \mathbb{R})$

$X = \{\text{lines through origin in } \mathbb{R}^3\}$ and so X is a homogeneous G-space.

hypothesis
mm

$X \cong G/_H$. $H \leq G$ is the stabilizer of your favorite point $p \in X$:

$H = \{ h \in G . hp = p\}$

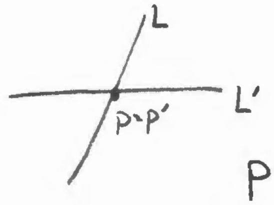
An invariant relation between points is a relation i.e. a subset $R \subseteq X \times X$ s.t. if $(p, q) \in R$ then $(gp, gq) \in R$ $\forall g \in G$ $\forall x, y \in X$

But the only invariant relations in this example are $p = q$ and $p \neq q$ since distance is not preserved by G.

More interestingly, let $Y = \{A \leq B .\ A$ is a 1dim subspace of $\mathbb{R}^3$, & B is a 2dim subspace of $\mathbb{R}^3$
$= \{\text{flags}\}$ where a flag is a point of $\mathbb{RP}^2$ on a line $L \subseteq \mathbb{RP}^2$

G acts transitively on Y (in fact, so does the Euclidean group) and there are various invariant relations between flags i.e. subsets $R \subseteq Y \times Y$ invariant under G

ex



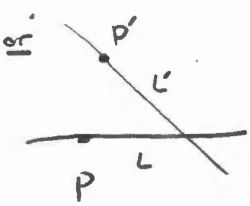
one invariant relation between (p, L) and (p', L')

$p = p'$ and $L \neq L'$

or

$p \neq p'$, $L = L'$

or

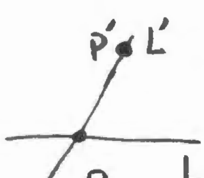


$p \neq p'$ and $L \neq L'$
$p \notin L'$ $p' \notin L$

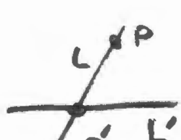
or

$p = p'$ $L = L'$

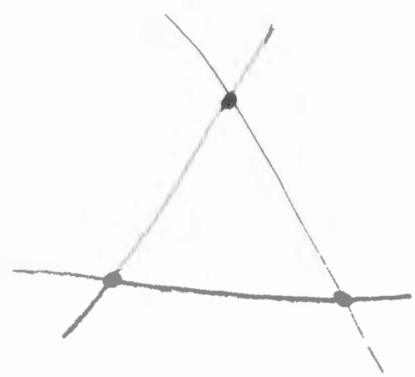
or $p \in L'$ but $L \neq L'$ and $p \neq p'$



or $p' \in L$ $L \neq L'$ $p \neq p'$



All relations are visible in this diagram of G-flags.



for any group G, we can make up a category $G\mathrm{Rel}$ where:

- Objects are G-Sets
- Morphisms are invariant relations.
  - Invariant relation $R: X \nrightarrow Y$ from the G-set X to the G-set Y is a relation (i.e. a subset $R \subseteq X \times Y$ s.t. $(x,y) \in R \Rightarrow (gx, gy) \in R \ \forall g \in G$ $\forall x, y \in X$

How do we compose morphisms?

Given any relation $R: X \nrightarrow Y$, $S: Y \nrightarrow Z$ we can compose them

to get $S \circ R: X \nrightarrow Z$

$$S \circ R = \{ (x,z) \in X \times Z : \exists y \in Y \text{ s.t. } (x,y) \in R, (y,z) \in Z \}$$

If R and S are invariant, so is $S \circ R$

There is a category Rel where:

- objects are sets,
- Morphisms are relations

Here $\hom(X,Y) = 2^{X \times Y}$ recall for any set S, 2^S is a complete atomic boolean algebra (CABA)

with $\subseteq$ as $\leq$, $\cap$ as $\wedge$ $:=(glb)$, $\cup : \vee$ $(= lub)$, compliments as $\neg$

So in Rel, $\hom(X,Y)$ is not merely a set, its a CABA.

The same is true for $G\mathrm{Rel}$: ex (if $R: X \nrightarrow Y$, $S \nrightarrow Y$ are invariant, so is $R \cap S$, $R \cup S$, R^c, and this obeys the laws of CABA)

In fact, Rel and $G\mathrm{Rel}$ are "CABA-enriched categories". What's an enriched category?

In category theory, we sometimes must over throw the tyranny of sets!

Instead of working in Sets all the time, we try to prove things that hold in many categori

But, the very definition of a category uses sets:

A category is a class of objects and for each pair of objects X, Y, a set $\hom(X,Y)$ and composition function

$$\circ : \hom(X,Y) \times \hom(Y,Z) \to \hom(X,Z) \quad \text{etc.}$$

The idea in enriched category theory is to generalize, replacing Set by some other category V and say:

A V-enriched category is a class of objects, and for each pair of objects X, Y, an object $\hom(X,Y) \in V$ and a composition morphism in V

$$\circ : \hom(X,Y) \otimes \hom(Y,Z) \to \hom(X,Z) \quad \text{(composition distributes over } \vee, \wedge)$$

etc.

Here we need V to be a "monoidal category" i.e. a category with some sort of "tensor product" $\otimes$ (Look for Kelly's Enriched Category theory)

It turns out that CABA's form a monoidal category, so it makes sense to talk about a CABA-enriched category, & Rel & GRel are such.

[Enriched Categories and internal Monoids]

A monoid is "the same" as a 1-object category: if you have a category $\mathbf{C}$ with one object X, there is a monoid $\hom(X,X)$ (set of all morphisms from X to X) with mult: $\circ : \hom(X,X) \times \hom(X,X) \to \hom(X,X)$

Conversely, given a monoid m, you can build a 1 object category X by letting $\hom(X,X) = m$, w. composition as mult.

More generally, suppose $\mathcal{V}$ is a monoidal category, i.e. a category with a tensor product: $\otimes : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ obeying certain rules.

Then recall a $\mathcal{V}$-enriched category, $\mathcal{C}$, has a class of objects and for any pair of objects $x, y \in \mathcal{C}$, a "hom-object" $\hom(x,y)\in\mathcal{V}$ and composition morphisms:

$$\circ : \hom(x,y)\otimes\hom(y,z)\to\hom(x,z)$$

A one object $\mathcal{V}$-enriched category is the same as a monoid internal to $\mathcal{V}$, or a monoid in $\mathcal{V}$, i.e. an object $m\in\mathcal{V}$ with a multiplication

$$m : m\otimes m\to m$$ that is associative and unital.

ex Suppose $\mathcal{V} = AbGp$ with the usual tensor product of abelian groups, then a monoid in $\mathcal{V}$ is called a ring!

It's an abelian group m, w a multiplication $m: m\otimes m\to m$ an abelian group homomorphism.

i.e. a function $m: m\times m\to m$ that's linear

ex if $\mathcal{V} = RMod$, for some commutative ring R, a monoid in $\mathcal{V}$ is called an R-algebra.

ex if $\mathcal{V} = Top$ w usual ~~product~~ $\times$ ~~of topological~~ spaces $\otimes$, a monoid in $\mathcal{V}$ is a topological monoid.

Back to our favorite example: klein geometry. Let G be a group, and let $GRel$ be the category w.

- G-Sets as objects
- G-invariant relations as morphisms.

This is a CABA-enriched category. So, if we take one object, i.e. one G-set X, we can form a 1-object CABA-enriched category w. X as the only object, $\hom(X,X)$ as the only homset, or "hom-CABA"

ex Projective plane geometry

$G: PGL(3,\mathbb{R})$ $\quad Y = \{\text{flags}\} = \left\{(P,L) : \begin{array}{l} P \subseteq \mathbb{R}^3 \text{ is a 1 dim subspace} \\ L \subseteq \mathbb{R}^3 \text{ is a 2 dim subspace} \end{array}, P \subseteq L \right\}$

$\hom(Y,Y)$ is a monoid in CABA. What is it like? Instead of describing everything, we'll just describe the atoms.

In general, given any group and any G-sets X, Y, what are the atoms in $\hom(X,Y)$? They're invariant relations $R : X \nrightarrow Y$ i.e.

$R \subseteq X \times Y$ and $(x,y) \in R \Rightarrow (gx, gy) \in R$

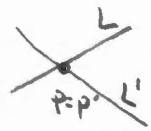
But they are the smallest nonempty subsets of this form.

So any atom R must contain a point (x,y) and thus all (gx, gy), $g \in G$

we get an orbit!

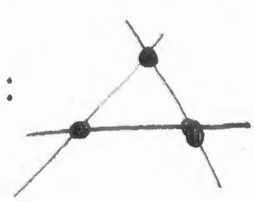
Thus, any orbit $\{(gx, gy) : g \in G\}$ is an atom in $\hom(X,Y)$

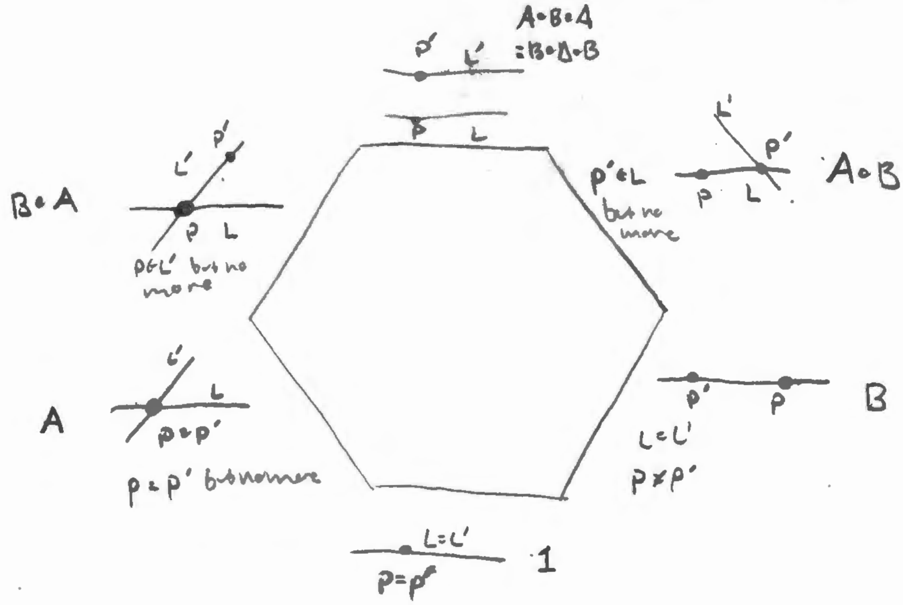
So if $G = PGL(3,\mathbb{R})$ then the atoms in $\hom(Y,Y)$ are the orbits of G acting on $Y \times Y$.

ex the orbit of this 

$x = (P,L)$
$y = (P',L')$

is the set of all pairs of flags sharing the same point and no more.

Last time we saw all 6 atoms in $\hom(Y,Y)$: 



The identity relation $1 \in \hom(Y,Y)$ is "two flags are the same"

Note: we can compose invariant relations and get invariant relations.

Let $A \in \hom(Y,Y)$ be the relation "having the same point but no more"

$$A \circ A = A \vee 1$$

If you change the line on a flag twice you could get the line

Let $B \in \hom(Y,Y)$ be "having the same ~~point~~ line but no more"

$$B \circ B = B \vee 1$$

$A \circ B =$ "$p' \in L$ but no more"

$B \circ A =$ "$p \in L'$ but no more"

$A \circ B \circ A = B \circ A \circ B =$ "nothing interesting"

in fact, this is a presentation for a monoid in CABA, $\hom(Y,Y)$

if we draw A as  B as 

then $A \circ B \circ A = B \circ A \circ B$ is called the "3rd Reidemeister move" or "Yang-Baxter eq."