Legible, decent notes for A+

Duality

Every category C has an opposite C^{op}.
The objects of C and C^{op} are the same, but C^{op} "turns around" the arrows of C.
So there's a 1-1 correspondence between morphisms in C and in C^{op}, with $f: x \to y$ in C corresponding to a morphism $f^{op}: y \to x$ in C^{op}.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & z \end{tikzcd}$$

Composing morphisms:

We compose morphisms in C^{op} by: $f^{op} \circ g^{op} = (g \circ f)^{op}$

$$\begin{tikzcd} & y \arrow[dr, "f^{op}"] & \\ z \arrow[ur, "g^{op}"] \arrow[rr, "f^{op} \circ g^{op}"'] & & x \end{tikzcd}$$

in C^{op}

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & z \end{tikzcd}$$

in C

The study of how categories C relate to their partners C^{op} is called duality. Should have $(C^{op})^{op} = C$. "Just like" for finite-dimensional vector spaces $(V^*)^* \cong V$

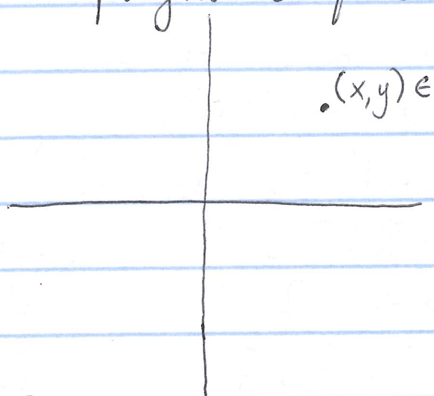
$\leftarrow$ natural isomorphism

It turns out that
"THE DUAL OF GEOMETRY IS ALGEBRA"

In geometry we study "points"; in algebra we study addition & multiplication (and fancier)

Greeks reasoned about algebra geometrically
Descartes realized a lot of geometry can reduce to algebra: this is called "analytic geometry".

We can associate to any finite-dimensional vector space V (over $\mathbb{R}$) a commutative ring $\mathcal{O}(V)$ consisting of all polynomial functions on V, with the usual $+$ and $\times$.



If $V = \mathbb{R}^n$, we have the algebra $\mathcal{O}(V)$ consisting of polynomials in the coordinate functions $x_1, \ldots, x_n$:

$$\mathcal{O}(V) = \underbrace{\mathbb{R}[x_1, \ldots, x_n]}_{\text{polynomials in } x_1, \ldots, x_n}$$

So we go from a "space" V (a bunch of points) to an algebra $\mathcal{O}(V)$.

Then we can describe certain subspaces of V:

$$X \xrightarrow[1\text{-}1]{} V$$

as quotient algebras (rings) of $\mathcal{O}(V)$:

$$\mathcal{O}(V) \xrightarrow[\text{onto}]{} \mathcal{O}(X) = \mathcal{O}(V)/I \quad \text{an ideal}$$

(This is why people got interested in ideals)

Example: The unit circle is a subspace of the plane:

$$S^1 \longrightarrow \mathbb{R}^2$$

$$S^1 = \{(x,y) : x^2+y^2-1=0\}$$

Then there is an algebra $\mathcal{O}(S^1)$ of polynomial functions on the unit circle, with

$$\mathcal{O}(S^1) = \mathbb{R}[x,y]/\langle x^2+y^2-1\rangle$$

the ideal generated by x^2+y^2-1

Thus the 1-1 map $S^1 \to \mathbb{R}^2$ gets turned around, giving

$$\mathcal{O}(\mathbb{R}^2) \longrightarrow \mathcal{O}(S^1)$$

which is just restriction: $f \in \mathcal{O}(\mathbb{R}^2)$ gives $f|_{S^1} \in \mathcal{O}(S^1)$.

Moreover, $f,g \in \mathcal{O}(\mathbb{R}^2)$ restrict to the same function on S^1

iff $\quad f-g \in \langle x^2+y^2-1\rangle$,

meaning $\quad f-g = (x^2+y^2-1)h \quad$ for some $h \in \mathcal{O}(\mathbb{R}^2)$

Algebraic geometry is the study of geometry using commutative rings.

Idea: Subspaces of V should correspond to quotient rings of $\mathcal{O}(V)$, or ideals of $\mathcal{O}(V)$.

Problems:

1) What about $\langle x^2+y^2+1\rangle \subseteq \mathcal{O}(\mathbb{R}^2)$?

What space corresponds here? The empty set?

A "circle of radius i"? That doesn't live in $\mathbb{R}^2$.

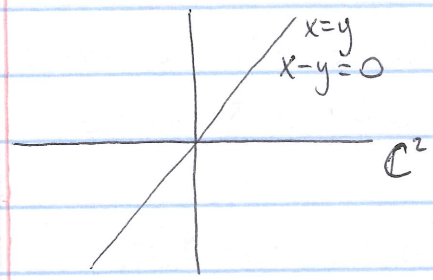
The function x^2+y^2+1 doesn't vanish on $\mathbb{R}^2$, so it seems the subspace of $\mathbb{R}^2$ corresponding to the ideal is $\emptyset$. But there is a simpler ideal that should correspond to $\emptyset$. Namely $\langle 1 \rangle$.

$$\emptyset = \{(x,y) : 1 = 0\}$$

We get two different ideals corresponding to same subspace.

One way out: Use $\mathbb{C}$ instead of $\mathbb{R}$.

Problem 2) Using $\mathbb{C}$ doesn't completely fix the issue of two different ideals corresponding to the same subspace.



There's a (complex) line in $\mathbb{C}^2$ given by $x=y$, with ideal $\langle x-y \rangle$

$\langle x-y \rangle \subseteq \mathbb{C}[x,y]$

But $(x-y)^2$ also vanishes only on this line, yielding a different ideal defining the same subspace $\langle (x-y)^2 \rangle \subseteq \mathbb{C}[x,y]$

Algebraic geometers found a workaround, but...

Grothendieck found a better solution. He figured out what we want to do.

~~Instead~~ He cut the Gordian knot & defined a new kind of space called an affine scheme such that the correspondence between algebra & geometry is perfect.

We're going to make up a category AffSch where the objects are "affine schemes" & morphisms are maps between them, such that $\mathrm{AffSch}^{op} = \mathrm{CommRing}$

What is AffSch? Take op of both sides:

$$(\mathrm{AffSch}^{op})^{op} = \mathrm{CommRing}^{op}$$

or $$\mathrm{AffSch} = \mathrm{CommRing}^{op}.$$

Example: the circle is an affine scheme; namely the commutative ring:

$$\mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle$$

The ~~real~~ plane is another example of an affine scheme: $\mathbb{Z}[x,y]$ $\cancel{\mathbb{R}[x,y]}$.

"The circle is included in the plane" means we have a homomorphism of commutative rings

$$\mathbb{Z}[x,y] \longrightarrow \mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle,$$ namely the quotient map. We also have

$$\mathbb{R}[x,y] \longrightarrow \mathbb{R}[x,y]/\langle x^2+y^2-1\rangle.$$

In "noncommutative geometry" we try to invent a weirder notion of "space" so that $\mathrm{AffSch} = \mathrm{CommRing}^{op}$ gets generalized to something $\underline{\qquad} = \mathrm{Ring}^{op}$