Category Theory Seminar

9/28/15

Learn the def. of

- category $\mathcal{C}$
- functor $f : \mathcal{C} \longrightarrow \mathcal{C}'$
- natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "f'"'{name=D}] & \mathcal{C}' \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

Duality

Every category $\mathcal{C}$ has an <u>opposite</u> category $\mathcal{C}^{op}$

$\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$,

but the morphisms are "turned around"

So there is a 1-1 correspondence between morphisms in $\mathcal{C}$ & morphisms in $\mathcal{C}^{op}$, with $f : x \longrightarrow y$ in $\mathcal{C}$ corresponding to a morphism $f^{op} : y \longrightarrow x$ in $\mathcal{C}^{op}$.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & z \end{tikzcd}$$

We compose morphisms in $\mathcal{C}^{op}$ by:

$f^{op} \circ g^{op} = (g \circ f)^{op}$

In $\mathcal{C}^{op}$:

$$\begin{tikzcd} & y \arrow[dr, "f^{op}"] & \\ z \arrow[ur, "g^{op}"] \arrow[rr, "f^{op}\circ g^{op} = (g\circ f)^{op}"'] & & x \end{tikzcd}$$

In $\mathcal{C}$:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & z \end{tikzcd}$$

The study of how categories $\mathcal{C}$ relate to their partners $\mathcal{C}^{op}$ is called <u>duality</u>.

$\boxed{\text{Note}}$ $(\mathcal{C}^{op})^{op} = \mathcal{C}$

'just like' for finite-dim vector spaces $(V^*)^* \cong V$

natural isomorphism

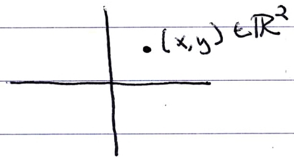
It turns out that the dual of geometry is algebra.

In geometry we study "points"; in algebra we study addition & multiplication.

Descartes realized we can reduce (a lot of) geometry to algebra:

this is called "analytic geometry"

We can associate to any finite-dimensional vector space V (over the real numbers) a commutative ring $\mathcal{O}(V)$ consisting of all polynomial functions on V, with usual addition & multiplication.



If $V = \mathbb{R}^n$, the algebra $\mathcal{O}(V)$ consists of polynomials in the coordinate functions $x_1, \dots, x_n$: $\mathcal{O}(V) = \underbrace{\mathbb{R}[x_1, \dots, x_n]}_{\text{polynomials in } x_1, \dots, x_n}$

So: we go from a "space" V (a bunch of points) to an algebra $\mathcal{O}(V)$.
Then we can describe certain subspaces of V:

$$X \xrightarrow[\text{1-1}]{} V$$

as quotient ~~algebras~~ rings of $\mathcal{O}(V)$:

$$\mathcal{O}(V) \xrightarrow[\text{onto}]{} \mathcal{O}(X) = \mathcal{O}(V)/I \leftarrow \text{an ideal}$$

$\boxed{\text{Example}}$ the unit circle is a subspace of the plane:

$$S^1 \longrightarrow \mathbb{R}^2$$

where $S^1 = \{(x,y) : x^2+y^2-1 = 0\}$

Then there is an algebra $\mathcal{O}(S^1)$ of polynomial functions on the unit circle, with $\mathcal{O}(S^1) = \mathbb{R}[x,y]/\langle x^2+y^2-1\rangle \leftarrow$ the ideal generated by x^2+y^2-1

So the 1-1 map $S^1 \longrightarrow \mathbb{R}^2$ gets turned around, giving $\mathcal{O}(\mathbb{R}^2) \longrightarrow \mathcal{O}(S^1)$ which is just restriction: $f \in \mathcal{O}(\mathbb{R}^2)$ gives $f|_{S^1} \in \mathcal{O}(S^1)$.

Moreover $f, g \in \mathcal{O}(V)$ restricted to the same function on S^1 iff

$f - g \in \langle x^2+y^2-1\rangle$

meaning $f - g = (x^2+y^2-1)h$ for some $h \in \mathcal{O}(V)$.

Algebraic geometry is the study of geometry using commutative rings.

Our idea is: subspaces of V should correspond to quotient rings of $\mathcal{O}(V)$, or ideal of $\mathcal{O}(V)$

Problems:

1) What about $\langle x^2+y^2+1\rangle \subseteq \mathcal{O}(\mathbb{R}^2)$

$$\| $$
$$\{(x^2+y^2+1)h : h \in \mathcal{O}(\mathbb{R}^2)\}$$

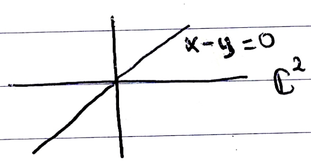
The function x^2+y^2+1 doesn't vanish on $\mathbb{R}^2$, so it seems the subspace of $\mathbb{R}^2$ corresponding to the ideal is $\emptyset$.

But there's another, simpler ideal that corresponds to $\emptyset \subseteq \mathbb{R}^2$.
Namely $\langle 1\rangle$.
$$\emptyset = \{(x,y) : 1=0\}$$

We're getting 2 ideals corresponding to the same subspace.

One way out: use $\mathbb{C}$ instead of $\mathbb{R}$.

2) Alas, using $\mathbb{C}$ doesn't completely fix the problem:
2 different ideals can correspond to the same subspace



There is a (complex) line in $\mathbb{C}^2$ given by $x=y$, with ideal
$\langle x-y\rangle \subseteq \mathbb{C}[x,y]$
But $(x-y)^2$ also vanishes only on this line, so we're getting a different ideal defining the same subspace $\langle (x-y)^2\rangle \subseteq \mathbb{C}[x,y]$

Algebraic geometers came up with a way around this...
but Grothendieck came along and found a better solution.

He cut the Gordian knot, and defined a new kind of space called an affine scheme such that the correspondence between algebra and geometry is perfect.

We're going to make up a category AffSch where objects are "affine schemes" and morphisms are maps between them, such that

$$\text{AffSch}^{op} = \text{CommRing}$$

What is AffSch?

Take op of both sides:

$$(\text{AffSch}^{op})^{op} = \text{CommRing}^{op}$$

OR $$\text{AffSch} = \text{CommRing}^{op}$$

Example the circle is an affine scheme, namely the comm. ring:

$$\mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle$$

The plane is an affine scheme, $\mathbb{Z}[x,y]$

"The circle is included in the plane" means we have a homomorphism of comm. rings

$$\mathbb{Z}[x,y] \longrightarrow \mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle$$

namely the quotient map.

We also have: $\mathbb{R}[x,y] \longrightarrow \mathbb{R}[x,y]/\langle x^2+y^2-1\rangle$

In "noncommutative geometry" we try to invent some new kind of "space" so that

$$\text{AffSch} = \text{CommRing}^{op}$$

gets generalized to something like

$$??? = \text{Ring}^{op}$$