10/5/2015

GEOMETRY	COMMUTATIVE ALGEBRA
Algebraic geometry: $C = $[affine schemes]	Ring Theory: $C^{op} = $[commutative rings]
Topology: $C = $[compact Hausdorff spaces]	C^*-algebra Theory: $C^{op} = $[commutative C^*-algebras]
Set theory: $C = $[sets]	Logic: $C^{op} = $[atomic Boolean algebras]

Look at CHaus = [compact Hausdorff spaces, continuous maps]

From a compact Hausdorff space X, an algebra

$$C(X) = \{f : X \to \mathbb{C} : f \text{ is continuous}\}.$$

This is an algebra with

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x),$$
$$(cf)(x) = cf(x) \quad c \in \mathbb{C}.$$

It's a commutative algebra. Better, it's a *-algebra with $(f^*)(x) = \overline{f(x)}$, meaning an algebra A with $*: A \to A$ such that

$$(f+g)^* = f^* + g^*$$
$$(fg)^* = g^* f^*$$
$$(cf)^* = \bar{c} f^*$$

Also, $C(X)$ has a norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

This makes sense since X is compact.

This makes $C(X)$ into a C^*-algebra, meaning that

$$\|fg\| \leq \|f\| \, \|g\|, \quad \|f^*\| = \|f\|, \text{ and}$$

"C^*-axiom": $\|f^* f\| = \|f\|^2$

$$\|f^*f\| = \sup_{x\in X} (f^*f)(x) = \sup_{x\in X} |f(x)|^2 = \left(\sup_{x\in X} |f(x)|\right)^2 = \|f\|^2.$$

So $C(X)$ is a commutative C^*-algebra. Next: morphisms.
Can we take a morphism $\phi: X \to Y$ in CHaus, i.e. a continuous map, and turn it into a (homo)morphism of commutative C^*-algebras?
A homomorphism between C^*-algebras, say $F: A \to B$, is a map such that

$$\left.\begin{aligned} F(a+b) &= \\ F(ab) &= \\ F(ca) &= \qquad\qquad c \in \mathbb{C} \\ F(a^*) &= \underbrace{\qquad\qquad}_{\text{The obvious things}} \end{aligned}\right.$$

$(F(a)+F(b),\ F(a)F(b),\ cF(a),\ F(a)^*)$

Assuming $\exists k>0$ such that $\|F(a)\| < k\|a\|$ $\forall a\in A$, all these imply $\|F(a)\| = \|a\|$.

So we get a category CommC*Alg = [Commutative C^*-algebras, C^*-algebra homomorphisms]

How does a continuous map $\phi: X \to Y$ between compact Hausdorff spaces give a C^*-algebra homomorphism between $C(X)$ & $C(Y)$?

We get one, $\phi^*: C(Y) \to C(X)$

by: $\phi^*(f)(x)$ $\qquad f \in C(Y),\ x \in X$

$= f(\phi(x))$

or: $\phi^*(f) = f\circ\phi$ $\qquad$ "ϕ pullback of f"

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "f\circ\phi"', "=\phi^* f"] & Y \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

This is why algebra is the "dual" of geometry

This is why algebra is the "dual" of geometry — it goes backwards: $\phi: X \to Y$ gives
$$\phi^*: C(Y) \to C(X)$$
Also $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ (check this)

So we're getting a functor:
$$C: \mathrm{CHaus} \longrightarrow \mathrm{Comm}\,C^*\mathrm{Alg}^{op}$$
$$X \longmapsto C(X)$$
$$\phi: X \to Y \longmapsto \phi^*: C(Y) \to C(X)$$

Gelfond–Naimark Thm: This functor is an equivalence of categories. I.e. There is a functor going back

$$\mathrm{Spec}: \mathrm{Comm}\,C^*\mathrm{Alg}^{op} \longrightarrow \mathrm{CHaus}$$
such that $\mathrm{Spec} \circ C \cong 1_{\mathrm{CHaus}}$
natural isomorphism

$$C \circ \mathrm{Spec} \cong 1_{\mathrm{Comm}\,C^*\mathrm{Alg}^{op}}$$

What is Spec? Given a commutative C^*-algebra A, how do we get a space $\mathrm{Spec}(A)$?
Let's start with $A = C(X)$. Then $\mathrm{Spec}(C(X))$ should give back X. How do we recover the points of X starting from $C(X)$? What's a point in X?
$\phi: \{*\} \longrightarrow X$, where $\{*\}$ is the one-point space.
I.e. given $x \in X$ $\exists$ map
$$\phi: \{*\} \longrightarrow X$$
$$* \longmapsto x$$
and conversely any map $\phi: \{x\} \to X$ determines a point in X.

Our functor $C : \mathrm{CHaus} \to \mathrm{Comm}\,C^*\mathrm{Alg}$ will turn

$\phi : \{*\} \to X$ into a homomorphism

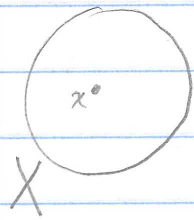
$$\phi^* : C(X) \to C(\{*\})$$
$$f \longmapsto f \circ \phi$$

In fact, $C(\{*\}) \cong \mathbb{C}$ where $g \in C(\{*\})$ gives $g(*) \in \mathbb{C}$.

So we get

$$\phi^* : C(X) \to C(\{*\}) \xrightarrow{\sim} \mathbb{C}$$
$$f \longmapsto f \circ \phi \longmapsto \underbrace{f \circ \phi(*)}_{= f(x)}$$

A point $x \in X$



gives $C(X) \to \mathbb{C}$
$f \longmapsto f(x)$ a homomorphism

Lemma – distinct point of X will give distinct homomorphisms $C(X) \to \mathbb{C}$

("There are enough continuous functions to separate points")

True because of the Hausdorffness

Lemma – any C^*-algebra homomorphism $\psi : C(X) \to \mathbb{C}$ comes from a point $x \in X$ via:

$$\psi(f) = f(x) \quad \forall f \in C(X)$$

So we get a 1-1 correspondence between points $x \in X$ and homomorphisms $\phi : C(X) \to \mathbb{C}$.

Given any commutative C^*-algebra A we define a set of points

$$\mathrm{Spec}(A) = \{\psi : A \to \mathbb{C} : \psi \text{ is a } C^*\text{-algebra homomorphism}\}$$

There is a topology making Spec(A) into a compact Hausdorff space. In this topology

ψ_i converges to ψ iff

$\psi_i(a)$ converges to $\psi(a)$ $\forall a \in A$.

Finally, given a C^*-algebra homomorphism $F: A \to B$, how do we get a map of spaces $\mathrm{Spec}(F): \mathrm{Spec}(B) \to \mathrm{Spec}(A)$?

$\psi \in \mathrm{Spec}(B)$ $\psi: B \to \mathbb{C}$ C^*-algebra homomorphism

$$\underbrace{\underbrace{\mathrm{Spec}(F)(\psi)}_{\text{in } \mathrm{Spec}(A)} \underbrace{(a)}_{\in A}}_{\in \mathbb{C}} = \underbrace{\psi(\underbrace{F(a)}_{\in B})}_{\in \mathbb{C}}$$

So we get functors

$$\begin{tikzcd} \mathrm{CHaus} \arrow[r, bend left, "C"] & \mathrm{CommC^*Alg}^{op} \arrow[l, bend left, "\mathrm{Spec}"] \end{tikzcd}$$

which are inverses (up to natural isomorphism)

Note:

$$\begin{tikzcd} \text{Spaces} \arrow[r, bend left, "\text{functions on the space}"] & \text{Commutative Rings} \arrow[l, bend left, "\text{spectrum}"] \end{tikzcd}$$

Point in a space $\rightleftarrows$ homomorphism from a commutative ring to a field

Subspace (inclusion $X \hookrightarrow Y$) monomorphism $\rightleftarrows$ Quotient ring or ideal (surjection or epimorphism $R \twoheadrightarrow$)