Geometry	Commutative Algebra
Algebraic geometry: C = [affine schemes]	Ring Theory C^{op} = [commutative rings]
Topology C = [compact Hausdorff spaces]	C^*-algebra Theory C^{op} = [commutative C^*-algebras]
Set theory C = [sets]	Logic C^{op} = [atomic Boolean algebras]

Look at CHaus = [compact Hausdorff space, continuous maps]

From a compact Hausdorff space X an algebra

$$C(X) = \{f : X \to \mathbb{C} : f \text{ is continuous}\}$$

This is an algebra with

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(cf)(x) = cf(x) \qquad c \in \mathbb{C}$$

It's a commutative algebra

It's a *-algebra with $(f^*)(x) = \overline{f(x)}$,

meaning an algebra A with $*: A \longrightarrow A$ s.t.

$$(f+g)^* = f^* + g^*$$
$$(fg)^* = g^* f^*$$
$$(cf)^* = \bar{c} f^*$$

Also $C(X)$ has a norm $\|f\| = \sup_{x \in X} |f(x)|$ This makes sense since X is compact.

This makes $C(X)$ into a C^*-algebra, meaning that:

$$\|fg\| \leq \|f\| \|g\|$$
$$\|f^*\| = \|f\|$$
$$\|f^* f\| = \|f\|^2 \qquad \text{"}C^*\text{-axiom"}$$

$$\|f^* f\| = \sup_{x \in X} (f^* f)(x)$$
$$= \sup_{x \in X} |f(x)|^2$$
$$= (\sup_{x \in X} |f(x)|)^2$$
$$= \|f\|^2$$

So $C(X)$ is a commutative C^*-alg.

Next: can we take a morphism $\varphi : X \to Y$ in CHaus, that is a cont. map, and turn it into a (homo) morphism of comm. C^*-algebras.

A homomorphism between C^*-algs, say $F : A \to B$, is a map s.t.

$F(a+b) = F(a) + F(b)$ $\quad a, b \in A$

$F(ab) = F(a)F(b)$

$F(ca) = cF(a)$ $\quad c \in \mathbb{C}$

$F(a^*) = F(a)^*$

$\exists\ K > 0$ s.t. $\|F(a)\| < K\|a\|$ $\forall\ a \in A$

All these imply $\|F(a)\| = \|a\|$

So we get a category

Comm C^* Alg = [comm. C^*-algebras, C^*-algebra homomorphisms]

How does a cont. map $\varphi : X \to Y$ between compact Hausdorff spaces give a C^*-alg. homo. between $C(X)$ and $C(Y)$?

We'll get one,

$\varphi^* : C(Y) \longrightarrow C(X)$

by: $\quad \varphi^*(f)(x) = f(\varphi(x))$ $\quad f \in C(Y),\quad x \in X$

or: $\quad \varphi^*(f) = f \circ \varphi$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend right, "f\circ\varphi"'] & Y \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

$\parallel$

$\varphi^* f$ $\quad$ "φ pullback of f"

This is why algebra is the "dual" of geometry - it goes backwards:

$\varphi : X \longrightarrow Y$ gives $\varphi^* : C(Y) \longrightarrow C(X)$

Also $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ (check this)

So we're getting a functor:

$$\begin{aligned} C: \mathrm{CHaus} &\longrightarrow \mathrm{Comm}\ C^*\mathrm{Alg}^{op} \\ X &\longmapsto C(X) \\ \varphi: X \to Y &\longmapsto \varphi^*: C(Y) \to C(X) \end{aligned}$$

Gelfand-Naimark Thm: This functor is an equivalence of categories.

I.e. there's a functor going back:

$$\mathrm{Spec}: \mathrm{Comm}\ C^*\mathrm{Alg}^{op} \longrightarrow \mathrm{CHaus}$$

s.t. $\mathrm{Spec} \circ C \cong 1_{\mathrm{CHaus}}$ and $C \circ \mathrm{Spec} \cong 1_{\mathrm{Com}\ C^*\mathrm{Alg}}$.

natural isomorphism

What's Spec?

Given a comm. C^*-alg. A, how do we get a space $\mathrm{Spec}(A)$?

Let's do $A = C(X)$.

Then $\mathrm{Spec}(C(X))$ should give back X.

How do we recover the points of X starting from $C(X)$?

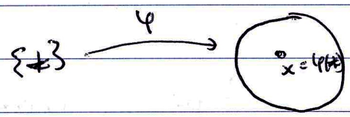
What's a point in X?

In terms of CHaus, what's a point of X?

It's a map $\varphi: \{*\} \longrightarrow X$ where $\{*\}$ is the one-point space.

i.e. given $x \in X$ there's a map

$$\begin{aligned} \varphi: \{*\} &\longrightarrow X \\ * &\longmapsto x \end{aligned}$$



& conversely any map $\varphi: \{*\} \longrightarrow X$ determines a point in X.

Our functor $C: \mathrm{CHaus} \to \mathrm{Comm}\ C^*\text{-alg}$ will turn $\varphi: \{*\} \to X$ into a homomorphism

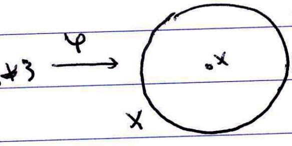
$$\begin{aligned} \varphi^*: C(X) &\longrightarrow C(\{*\}) \\ f &\longmapsto f \circ \varphi \end{aligned}$$

In fact $C(\{*\}) \cong \mathbb{C}$ where $g \in C(\{*\})$ gives $g(*) \in \mathbb{C}$

So we get

$$\begin{aligned} \varphi^*: C(X) &\longrightarrow C(\{*\}) \xrightarrow{\sim} \mathbb{C} \\ f &\longmapsto f \circ \varphi \longmapsto f \circ \varphi(*) \\ & \qquad\qquad\qquad\quad\ \ \, \shortparallel \\ & \qquad\qquad\qquad\quad\ \ f(x) \end{aligned}$$

A point $x \in X$



gives a homorphism $C(X) \longrightarrow \mathbb{C}$

$f \longmapsto f(x)$

In short: any point $x \in X$ gives a homorphism from $C(X)$ to $\mathbb{C}$ called evaluation at x.

! pt $\forall$ hom.

$\boxed{\text{Lemma}}$ Distinct points of X give distinct homomorphisms $C(X) \to \mathbb{C}$.

("There are enough continuous functions to separate points" for a compact Hausdorff space)
$\to$ Stone – Weierstrass Theorem

$\exists$ pt $\forall$ hom.

$\boxed{\text{Lemma}}$ Any C^*-alg. homomorphism $\Psi : C(X) \to \mathbb{C}$ comes from a point $x \in X$ via: $\Psi(f) = f(x)$ $\forall f \in C(X)$.

So we get a 1-1 correspondence between points $x \in X$ and homomorphisms $\Psi : C(X) \to \mathbb{C}$.

So given any comm. C^*-algebra A we define a set of points
$\mathrm{Spec}(A) = \{\Psi : A \to \mathbb{C} : \Psi \text{ is a } C^*\text{-alg. homomorphism}\}$

There's a topology making $\mathrm{Spec}(A)$ into a compact Hausdorff space.
In this topology Ψ_i converges to Ψ iff $\Psi_i(a) \to \Psi(a)$ for all $a \in A$

Finally, given a C^*-alg. homo. $F : A \to B$, how do we get a map of spaces $\mathrm{Spec}(F) : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$?

$\mathrm{Spec}(F)(\Psi)(a) = \Psi(F(a))$

$\Psi : B \to \mathbb{C}$ C^*-alg homo.

$a \in A$

$F(a) \in B$

$\Psi(F(a)) \in \mathbb{C}$

So we get functors

$$\begin{tikzcd} \text{CHaus} \arrow[r, bend left, "C"] & \text{Comm. } C^*\text{Alg}^{op} \arrow[l, bend left, "\text{Spec}"] \end{tikzcd}$$

which are inverses (up to nat. iso.)

$\boxed{\text{Note}}$

$$\begin{tikzcd} \text{Spaces} \arrow[r, bend left, "\text{functions on the space}"] & \text{Comm Rings} \arrow[l, bend left, "\text{spectrum}"] \end{tikzcd}$$

$$\begin{tikzcd} \text{Point in a space} \arrow[r, bend left] & \text{Homomorphism from a comm. ring to a field} \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \text{Subspace} \arrow[r, bend left] & \text{Quotient ring or ideal} \arrow[l, bend left] \end{tikzcd}$$

(inclusion $X \hookrightarrow Y$) or monomorphism

(surjection or epimorphism $R \twoheadrightarrow S$)