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The duality between set theory & logic

Idea:

Subset (of X) operations	Logic operations
$\cup$	$\vee$ (or)
$\cap$	$\wedge$ (and)
$\emptyset$	F (false)
X	T (true)
^{c} (complement)	$\neg$ (not)

e.g. $x \in S \cup T \iff x \in S \vee x \in T$

Last time we saw that given a compact Hausdorff space X, we get a commutative C^*-algebra

$$C(X) = \hom_{Top}(X, \mathbb{C}) \quad \text{(continuous maps from } X \text{ to } \mathbb{C}\text{)}$$

and, given a commutative C^*-algebra A, we get a compact Hausdorff space

$$\mathrm{Spec}(A) = \hom_{CommC^*Alg}(A, \mathbb{C}) \quad \text{(homomorphisms of } C^*\text{-algebras)}$$

We call $\mathbb{C}$ here a dualizing object: "the same object x in two different categories $\mathcal{C}$ & $\mathcal{D}$" such that

$$\mathcal{C} \longrightarrow \mathcal{D}^{op}$$

for some mysterious reason...

$$c \longmapsto \hom_{\mathcal{C}}(c, x) \in \mathcal{D}$$

and

$$\mathcal{D}^{op} \longrightarrow \mathcal{C}$$

$$d \longmapsto \hom_{\mathcal{D}}(d, x) \in \mathcal{C}$$

are inverse functors, so $\mathcal{C}$ & $\mathcal{D}^{op}$ are equivalent.

Let's make the mystery work again!

We'll relate sets & Boolean algebras using the dualizing object

$$2 = \{0,1\} \cong \{F, T\}.$$

Boolean algebras are a little bit like C^*-algebras, but:

Commutative C^*-algebras	Boolean algebras
$+$	$\vee$
$\cdot$	$\wedge$
0	F
1	T
$\mathbb{C}$	$2 = \{F, T\}$

Given a set X, we'll make a Boolean algebra

$$2^X = \hom_{Set}(X, 2).$$

An element here is a function $f: X \to \{0,1\}$. Any subset $S \subseteq X$ gives a characteristic function:

$$\chi_S(x) = \begin{cases} 0 & x \notin S \\ 1 & x \in S \end{cases}$$

and conversely any such function gives a subset of X, so 2^X is just another way to think of the power set of X.

The operations $\cup, \cap, {}^c$ on subsets of X correspond to operations $\vee, \wedge, \neg$ on functions $f: X \to \{0,1\}$

$$\chi_{S \cup T} = \chi_S \vee \chi_T \quad \text{where} \quad \chi_S \vee \chi_T = \chi_S(x) \vee \chi_T(x) \quad \text{etc.}$$

$(2^S, \vee, \wedge, 0, 1)$ will be a Boolean algebra.

~~De~~ Note: $S \subseteq T \iff \chi_S \le \chi_T$

i.e. if $\chi_S(x) = 1$ then $\chi_T(x) = 1$.

Note: "$\le$" is not a separate concept:

$$\chi_S \le \chi_T \iff \chi_S \wedge \chi_T = \chi_S$$
$$\iff \chi_S \vee \chi_T = \chi_T$$

Def: A partially ordered set $(A, \le)$ is called a <u>lattice</u> if every pair $a, b \in A$ has a least upper bound $a \vee b$ and greatest lower bound $a \wedge b$, and also a least element $0 = F$ and a greatest element $1 = T$

Def: A <u>distributive lattice</u> is a lattice where $\wedge$ & $\vee$ distribute over each other.

Def: A <u>Boolean algebra</u> is a distributive lattice where every element $x \in A$ has a <u>complement</u> $\neg x$ such that

$$x \wedge \neg x = F$$
$$x \vee \neg x = T.$$

(If a complement exists, it is unique)

Example: For any set S, 2^S is a Boolean algebra with pointwise defined $\le$; given $f, g \in 2^S$ we say

$$f \le g \text{ if } f(x) \le g(x) \ \forall x \in S.$$

It thus has pointwise defined $\vee, \wedge, 0, 1, \neg$.

E.g. $(\neg f)(x) = \neg f(x)$

Not every Boolean algebra is isomorphic to one of this form!
The Boolean algebras of the form 2^S are "complete atomic Boolean Algebras".

Def: A complete Boolean algebra A is one where every subset $S \subseteq A$ has a least upper bound $\bigvee_{x \in S} x$ and greatest lower bound $\bigwedge_{x \in S} x$, and these distribute over each other.

Def: An atom in a Boolean algebra A is an element $x \in A$ such that $x \neq 0$ and if $y < x$ then $y = 0$.

Example: in 2^S the atoms correspond to the elements of S, or singletons $\{s\} \subseteq S$.

Def: A Boolean algebra is atomic if $\forall x \in A$, $x = \bigvee_{\lambda \in \Lambda} y_\lambda$ where $y_\lambda \in A$ are atoms.

There is a category CABA of complete atomic Boolean algebras and homomorphisms of complete Boolean algebras:

$\phi: A \to B$ preserving $\vee, \wedge, 0, 1, \neg, \bigvee, \bigwedge$

enough to preserve these two

There is a category Set of sets & functions.

Thm Set is equivalent to $CABA^{op}$ via functors

$$\begin{aligned} Set &\longrightarrow CABA^{op} \\ S &\longmapsto 2^S = \hom_{Set}(S, 2) \end{aligned}$$

and

$$\begin{aligned} CABA^{op} &\longrightarrow Set \\ A &\longmapsto \hom_{CABA}(A, 2), \end{aligned}$$

where $2 = \{0, 1\}$ is a CABA in the obvious way.

Given a function $\phi : S \to T$ we get a complete Boolean algebra homomorphism

$$\phi^* : 2^T \to 2^S \quad \text{by:}$$

$$\underbrace{\phi^*(f)(s)}_{\in 2} = f(\phi(s))$$

$f \in 2^T$
$s \in S$

This is called a pullback:

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[dr, "\phi^* f = f \circ \phi"'] & T \arrow[d, "f"] \\
& 2
\end{tikzcd}$$

measurable subsets of $\mathbb{R}$

atomic but not complete Boolean algebra under pointwise operations

$$\{ f \in L^\infty[0,1] : f(x) = \overset{\text{F or T}}{0 \text{ or } 1} \quad \forall x \in X \}$$

Complete but not atomic Boolean algebra under pointwise operations.