10/12/15

The duality between set theory & logic.

The basic idea:

operations on subsets of X	logical operations
$\cup$	$\vee$ (or)
$\cap$	$\wedge$ (and)
$\emptyset$	$F = 0$
X	$T = 1$
complement	$\neg$ (not)

e.g. $x \in S \cup T \iff x \in S$ or $x \in T$
etc.

Let's fit this into the same mold as last time.
We saw that given a compact Hausdorff space X, we get a commutative C^*-alg.
$C(X) = \hom_{Top}(X, \mathbb{C})$ (continuous maps from X to $\mathbb{C}$)
Also, given a comm. C^*-algebra A, we get a compact Hausdorff space
$\mathrm{Spec}(A) = \hom_{\mathrm{Comm}\, C^*\mathrm{Alg}}(A, \mathbb{C})$ (homomorphisms of C^*-algebras)

We call $\mathbb{C}$ here a dualizing object:
"the same object x in 2 different categories $\mathcal{C}$ & $\mathcal{D}$" such that

$\mathcal{C} \longrightarrow \mathcal{D}^{op}$
$c \longmapsto \hom_{\mathcal{C}}(c, x) \in \mathcal{D}$ ← for some mysterious reason...

&

$\mathcal{D}^{op} \longrightarrow \mathcal{C}$
$d \longmapsto \hom_{\mathcal{D}}(d, x) \in \mathcal{C}$ ← for some mysterious reason...

are inverses,
so $\mathcal{C}$ & $\mathcal{D}^{op}$ are equivalent.

Similarly we'll relate sets & Boolean algebras using the dualizing object
$2 = \{0, 1\} \cong \{F, T\}$

Boolean algebras are a little bit like C^*-algebras, but:

Comm. C^*-algebras	Boolean algebras
$\mathbb{C}$	$2 = \{F, T\}$
$+$	$\vee$
$\cdot$	$\wedge$
0	F
1	T

So, given a set X, we'll make a Boolean algebra
$$2^X = \hom_{Set}(X, 2)$$
An element here is a fn. $f: X \longrightarrow \{0,1\}$.
Any subset $S \subseteq X$ gives such a fn:
$$\chi_S(x) = \begin{cases} 0 & x \notin S \\ 1 & x \in S \end{cases}$$
& conversely any such function gives a subset of X, so 2^X is just another way to think of the power set of X.

The operations $\cup, \cap, {}^c$ (complement) on subsets of X corr. to operations $\vee, \wedge, \neg$ on functions $f: X \longrightarrow \{0,1\}$
$$\chi_{S \cup T} = \chi_S \vee \chi_T$$
where $\chi_S \vee \chi_T(x) = \chi_S(x) \vee \chi_T(x)$
& so on.

$(2^S, \vee, \wedge, 0, 1)$ will be a Boolean algebra:

Note $S \subseteq T \iff \chi_S \leq \chi_T$
i.e. if $\chi_S(x) = 1$ then $\chi_T(x) = 1$
Note "$\leq$" is not a separate concept:
$$\chi_S \leq \chi_T \iff \chi_S \wedge \chi_T = \chi_S$$
$$\iff \chi_S \vee \chi_T = \chi_T$$

Def A partially ordered set $(A, \leq)$ is called a lattice if every pair $a, b \in A$ has a least upper bound $a \vee b$ & greatest lower bound $a \wedge b$, and also a least element $0 = F$ and a greatest element $1 = T$.

Def A distributive lattice is one where $\wedge$ & $\vee$ distribute over each other.

Def A Boolean algebra is a distributive lattice A where every element $x \in A$ has a complement $\neg x$ such that

$$x \wedge \neg x = F \qquad x \vee \neg x = T$$

(If a complement exists, it's unique.)

Ex For any set S, 2^S is a Boolean algebra with pointwise defined $\leq$:
given $f, g \in 2^S$ we say $f \leq g$ if $f(x) \leq g(x) \quad \forall x \in S$
It thus has pointwise defined $\vee, \wedge, 0, 1, \& \neg$,
e.g. $(\neg f)(x) = \neg f(x)$

Alas, not every Boolean algebra is isomorphic to one of this form!

The Boolean algebras of the form 2^S are "complete atomic Boolean algebras".

Def A complete Boolean alg. A is one where every subset $S \subseteq A$ has a l.u.b. $\bigvee_{x \in S} x$ and g.l.b. $\bigwedge_{x \in S} x$, and these distribute over each other.

Def An atom in a Boolean algebra A is an element $x \in A$ s.t. $x \neq 0$ and if $y < x$ then $y = 0$.

Ex In 2^S the atoms correspond to the elements of S, or singletons $\{s\} \subseteq S$.

Def A Boolean algebra A is atomic if $\forall x \in A$, $x = \bigvee_{\lambda \in \Lambda} y_\lambda$ where $y_\lambda \in A$ are atoms.

There's a category CABA of complete atomic Boolean algebras & <u>homomorphisms</u> of complete Boolean algebras:

$\varphi : A \longrightarrow B$ preserving $\vee, \wedge, 0, 1, \neg, \bigvee, \bigwedge$

There's a category Set of sets and functions.

$\boxed{\text{Thm}}$ Set is equivalent to CABA^{op} via these functors

$$\mathrm{Set} \longrightarrow \mathrm{CABA}^{op}$$
$$S \longmapsto 2^S = \hom_{\mathrm{Set}}(S, 2)$$

and

$$\mathrm{CABA}^{op} \longrightarrow \mathrm{Set}$$
$$A \longmapsto \hom_{\mathrm{CABA}}(A, 2)$$

where $2 = \{F, T\}$ is a CABA in the obvious way.

Given a function $\varphi : S \longrightarrow T$ we get a complete Boolean alg. homo.

$\varphi^*(f)(s) = f(\varphi(s))$ $\quad f \in 2^T$ i.e. $f \in T \longrightarrow 2$ $\quad s \in S$

This is called a pullback:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[drr, bend right, "\varphi^* f = f \circ \varphi"'] & T \arrow[dr, "f"] & \\ & & 2 \end{tikzcd}$$

$\boxed{\text{Ex}}$ $\{f \in L^\infty[0,1] : f(x) = \overset{F}{0} \text{ or } \overset{T}{1} \text{ for all } x \in X\}$

This is a complete but not atomic Boolean algebra under pointwise operations.

(There are no atoms)