10/19/15

GEOMETRY	ALGEBRA
Algebraic Geometry (affine schemes)	Commutative Rings
Topology (compact Hausdorff spaces)	Commutative C^*-algebras
Set Theory	Complete atomic Boolean algebras
?	Boolean algebras
?	Linear algebra (finite-dimensional vector spaces)
?	Finite abelian groups
	Propositional Logic

The opposite of the category of <u>all</u> Boolean algebras is the category of <u>Stone spaces</u>: compact Hausdorff spaces that are <u>totally disconnected</u>: every open set is closed (& vice versa).
The Boolean algebra of a Stone space X consists of its open subsets, with $A \cup B$ as "$\vee$", $A \cap B$ as "$\wedge$", A^c as "$\neg$"

Let FinVect be the category of finite-dimensional vector spaces over your favorite field (e.g. $\mathbb{R}$) & linear maps.
What is $\mathrm{FinVect}^{op}$? A typical morphism in FinVect is
$T: \mathbb{R}^n \to \mathbb{R}^m$.
A morphism in $\mathrm{FinVect}^{op}$ is thus $T^{op}: \mathbb{R}^m \to \mathbb{R}^n$ ....
suspiciously similar to the transpose $T^t: \mathbb{R}^m \to \mathbb{R}^n$ in FinVect.
In fact, $\mathrm{FinVect}^{op} \cong \mathrm{FinVect}$, with

$$\underset{\text{in FinVect}}{T: \mathbb{R}^n \to \mathbb{R}^m} \longmapsto T^t: \mathbb{R}^m \to \mathbb{R}^n \text{ (in FinVect)}$$

i.e. $(T^t)^{op}: \mathbb{R}^n \to \mathbb{R}^m$ in $\mathrm{FinVect}^{op}$

We can also get the equivalence $\mathrm{FinVect} \cong \mathrm{FinVect}^{op}$ using $\mathbb{R} \in \mathrm{FinVect}$ as the dualizing object:

$$\mathrm{FinVect} \longrightarrow \mathrm{FinVect}^{op}$$
$$V \longmapsto \hom(V, \mathbb{R}) = V^*$$
$$T: V \to W \longmapsto T^*: W^* \to V^* \quad \text{in FinVect}$$
$$\text{or } (T^*)^{op}: V^* \to W^* \quad \text{in } \mathrm{FinVect}^{op}$$

So FinVect straddles the worlds of geometry and algebra, being its own opposite.

Also, the category [finite abelian groups, group homomorphisms] is its own "op" (but for different reasons)

In both cases you can break everything down into direct sums of building blocks

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Change of theme

Galois Theory

Galois theory is secretly about dualities between posets.

Def: A <u>poset</u> is a partially ordered set $(S, \leq)$ where $\leq$ is reflexive, transitive & antisymmetric: $x \leq y$ & $y \leq x \Rightarrow x = y$

If $(S, \leq)$ is a poset, we get a category with elements of S as objects & there exists a unique morphism $f: x \to y$ iff $x \leq y$ $(x, y \in S)$, and no morphisms $f: x \to y$ otherwise.

In fact, the categories we get this way are precisely those with:

1) at most 1 morphism from any object x to any object y
2) if there are morphisms $f: x \to y$ & $g: y \to x$, then $x = y$.

So to a category theorist, a poset is a category with these two properties.

Given categories of this kind, a functor is really just an <u>order-preserving map</u> ~~f:(S,≤)→(T,~~

$$f: (S, \leq) \to (T, \leq),$$

i.e. a function such that

$$x \leq y \text{ in } S \Rightarrow f(x) \leq f(y) \text{ in } T.$$

Given a ~~g~~ category of this sort coming from $(S, \leq)$, its opposite comes from the poset $(S, \leq^{op})$, where

$$x \leq^{op} y \quad \text{~~means~~ iff} \quad y \leq x.$$

We'll write $x \geq y$ for $x \leq^{op} y$.

What are adjoint functors between categories of this sort?

Def: Given categories C, D, we say a functor $L: C \to D$ is the <u>left adjoint</u> of a functor $R: D \to C$, or ~~R:D~~ $R: D \to C$ is the <u>right adjoint</u> of L, if there is a natural 1-1 correspondence

$$\hom_D(Lx, y) \cong \hom_C(x, Ry) \qquad \forall x \in C, y \in D.$$

Example: Let $L: \mathrm{Set} \to \mathrm{Grp}$ (L = libre = freedom)
send any set S to the free group on S,
and $R: \mathrm{Grp} \to \mathrm{Set}$
send any group G to its underlying set.

Here $\cancel{\#}$ $\hom_{\mathrm{Grp}}(LS, G) \cong \hom_{\mathrm{Set}}(S, RG)$

What does this look like for posets?
Example: what are adjoint functors between posets $(S, \leq)$ and $(T, \leq)$?
It's a pair of order-preserving functions

$$\begin{tikzcd} (S,\leq) \arrow[r, bend left, "L"] & (T,\leq) \arrow[l, bend left, "R"] \end{tikzcd}$$

such that $Lx \leq y \iff x \leq Ry$.
This comes from $\hom_D(Lx, y) \cong \hom_C(x, Ry)$.

Def: A pair of adjoint functors between posets is called a <u>Galois correspondence</u>.

Thm: Suppose $\begin{tikzcd} (S,\leq) \arrow[r, bend left, "L"] & (T,\leq) \arrow[l, bend left, "R"] \end{tikzcd}$ is a Galois correspondence.

Then we get an order-preserving map $RL: (S, \leq) \to (S, \leq)$.
Let's write $\bar{x}$ for RLx. Then: $x \leq \bar{x}$ $\forall x \in S$
and $\bar{\bar{x}} = \bar{x}$ $\forall x \in S$

$$Lx \leq Lx \Rightarrow x \leq RLx$$

So we say $\bar{\ }$ is a <u>closure operator</u> on the poset $(S, \leq)$

Similarly, write y° for LRy.

Then $y^\circ \leq y \quad \forall y \in T$

and $(y^\circ)^\circ = y^\circ \quad \forall y \in T$.

So $^\circ$ behaves like the "interior" operation on subsets of a topological space – it's a closure operator on $(T, \leq)^{op}$.

Finally, L & R give a bijection between <u>closed</u> elements of S (meaning $x \in S$ with $\bar{x} = x$) and <u>open</u> elements of T (meaning $y \in T$ such that $y^\circ = y$).

Later, we will replace T by T^{op} to make this "Finally" bit look nicer.