The opposite of the category of all Boolean algebras is the category of Stone spaces: compact Hausdorff spaces that are totally disconnected, in which every open set is closed (& vice versa).

The Boolean algebra of a Stone space X consists of its open subsets, with $A \cup B$ as "$\vee$", $A \cap B$ as "$\wedge$", A^c as "$\neg$".

Let FinVect be the category of finite-dimensional vector spaces over your favorite field & linear maps. What's FinVect^{op}? A typical morphism in FinVect is $T: \mathbb{R}^n \to \mathbb{R}^m$, so a morphism in FinVect^{op} is $T^{op}: \mathbb{R}^m \to \mathbb{R}^n$. This is suspiciously similar to the transpose $T^t: \mathbb{R}^m \to \mathbb{R}^n$ in FinVect. In fact, $\text{FinVect} \xrightarrow{\sim} \text{FinVect}^{op}$ with:

$$\underset{\text{in FinVect}}{T: \mathbb{R}^n \to \mathbb{R}^m} \quad \mapsto \quad \underset{\text{in FinVect}^{op}}{(T^t)^{op}: \mathbb{R}^n \to \mathbb{R}^m} \quad \text{or} \quad \underset{\text{in FinVect}}{T^t: \mathbb{R}^m \to \mathbb{R}^n}$$

We can also get the equivalence of FinVect & FinVect^{op} using $\mathbb{R} \in \text{FinVect}$ as the dualizing object:

$$\begin{aligned} \text{FinVect} &\longrightarrow \text{FinVect}^{op} \\ V &\mapsto \text{Hom}(V, \mathbb{R}) = V^* \\ T: V \to W &\mapsto \underset{\text{in FinVect}}{T^*: W^* \to V^*} \quad \text{or} \quad \underset{\text{in FinVect}^{op}}{(T^*)^{op}: V^* \to W^*} \end{aligned}$$

So, FinVect straddles the worlds of geometry & algebra by being its own opposite.

Also, the category of finite abelian groups & group homomorphisms is its own opposite.

Galois Theory

Galois Theory is secretly about dualities between posets (partially ordered sets).

If $(S, \leq)$ is a poset, then we get a category with elements of S as objects & there exists a unique morphism $f: x \to y$ iff $x \leq y$ for $x, y \in S$, & no morphisms otherwise.

The categories that we get in this way are precisely those with:

1) at most one morphism from any object x to any object y

2) if there are morphisms $f: x \to y$ & $g: y \to x$, then $x = y$.

So to a category theorist, a poset is a category with these two properties.

Given categories of this kind, a functor is really just an order-preserving map $f: (S, \leq) \to (T, \leq)$.

Given a category of this sort coming from the poset $(S, \leq)$, its opposite comes from the poset $(S, \leq^{op})$ where $x \leq^{op} y$ iff $x \geq y$.

What are adjoint functors between categories of this sort?

Defn: Given categories C, D, we say a functor $L: C \to D$ is *left adjoint* of a functor $R: D \to C$, or R is *right adjoint* of L, if there's a natural 1-1 correspondence $\mathrm{Hom}_D(Lx, y) \cong \mathrm{Hom}_C(x, Ry)$ $\forall\ x \in C, y \in D$.

Example: Let $L: \mathrm{Set} \to \mathrm{Grp}$ send any set S to the free group on S, & $R: \mathrm{Grp} \to \mathrm{Set}$ send any group G to its underlying set.
[L = liberty! = freedom]
Here, $\mathrm{Hom}_{\mathrm{Grp}}(LS, G) \cong \mathrm{Hom}_{\mathrm{Set}}(S, RG)$.

Example: What are adjoint functors between posets $(S, \leq)$ & $(T, \leq)$?
It's a pair of order-preserving functions $(S, \leq) \underset{R}{\overset{L}{\rightleftarrows}} (T, \leq)$ such that $Lx \leq y$ iff $x \leq Ry$, which comes from $\mathrm{Hom}_T(Lx, y) \cong \mathrm{Hom}_S(x, Ry)$.

Defn: A pair of adjoint functors between posets is called a *Galois correspondence*.

Thm: Suppose $(S, \leq) \underset{R}{\overset{L}{\rightleftarrows}} (T, \leq)$ is a Galois correspondence. Then we get an order preserving map $RL: (S, \leq) \to (S, \leq)$. Let's write $\bar{x}$ for RLx. Then $x \leq \bar{x}$ $\forall\ x \in S$ (because $Lx \leq Lx \Rightarrow x \leq RLx$) & $\bar{\bar{x}} = \bar{x}$ $\forall\ x \in S$. So we say $\bar{\ }$ is a *closure operator* on the poset $(S, \leq)$. Similarly, write y° for LRy. Then $y^\circ \leq y$ $\forall\ y \in T$ & $(y^\circ)^\circ = y^\circ$ $\forall\ y \in T$. So $^\circ$ behaves like the "interior" operation on subsets of a top. space – it's a closure operator on $(T, \leq)^{op}$. Finally, L & R give a bijection between *closed* elements of S (meaning $x \in S$ w/ $x = \bar{x}$) & *open* elements of T (meaning $y \in T$ w/ $y^\circ = y$).