11/2/15

Groupoids

Def: A morphism $f: x \to y$ in a category has an <u>inverse</u> $g: y \to x$ if $fg = 1_y$ & $gf = 1_x$. If f has an inverse, it is unique so we write it as f^{-1}.
A morphism with an inverse is called an <u>isomorphism</u>.
If there is an isomorphism $f: x \to y$ we say x & y are <u>isomorphic</u>. It is more useful to have an isomorphism than merely to know things are isomorphic.

Def: A <u>groupoid</u> is a category where all morphisms are isomorphisms.

E.g. Any group G gives a groupoid with one object, $*$, and morphisms $g: * \to *$ corresponding to elements $g \in G$, with composition coming from multiplication in G.

Conversely, any 1-object groupoid gives a group.
So "a group is a 1-object groupoid".

More generally, if C is any category and $x \in C$, notation abuse: x is an object in C
the isomorphisms $f: x \to x$ form a group under composition, called the <u>automorphism group</u> $\mathrm{Aut}(x)$

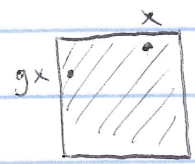
E.g. $\mathrm{Aut}(\square) \cong \mathbb{Z}_4$ (rotational symmetries)

Example: Given any category C there is a groupoid, the <u>core</u> of C, denoted C_0, whose objects are those of C & whose morphisms are the isomorphisms of C, composed as before. Would need to check composition of isomorphisms still give isomorphisms.

Example: if $\quad \text{FinSet} = [\text{finite sets, functions}]$
then $\quad \text{FinSet}_0 = [\text{finite sets, bijections}]$,
and if n is your favorite n-element set, $\text{Aut}(n) = S_n$
the symmetric group. If you understand S_n, you understand most of what goes on in FinSet_0.
FinSet_0 "unifies" all the symmetric groups.

Example: Suppose G is a group acting on a set X:

$$\alpha : G \times X \longrightarrow X$$
$$(g, x) \longmapsto gx$$



Often people form the set X/G, the quotient set where an element $[x]$ is an equivalence class of elements $x \in X$ where $x \sim y$ iff $y = gx$ $\exists g \in G$. (the orbit).
But a "better" thing to do is form the translation groupoid $X /\!/ G$, where:

- objects are elements $x \in X$
- a morphism from x to y is a pair (g, x) where $g \in G$ and $gx = y$.

$$x \xrightarrow{(g,x)} y$$

- The composite of $x \xrightarrow{(g,x)} y$ and $y \xrightarrow{(h,y)} z$ is $x \xrightarrow{(hg,x)} z$

In X/G we say x & y are equal if $gx = y$;
in $X /\!/ G$ we say they are isomorphic, or more precisely, we have a chosen isomorphism $(g, x) : x \to y$

To a (pretty good) first approximation, a "moduli space" is a set X/G, given some obvious topology, while a "moduli stack" is a group $X//G$, where the sets of objects and morphisms have topologies.

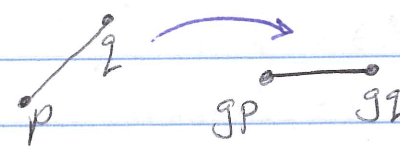
Example: Let X be the set of line segments in the Euclidean plane.

Let G be the ~~group of~~ Euclidean group of the plane: g preserves distances

all bijections $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|gp - gq| = |p - q|$

i.e. g preserves distances. This causes angles to be preserved

G acts on X:



More precisely, $X = \mathbb{R}^2 \times \mathbb{R}^2$ & G acts on it via

$g(p,q) = (gp, gq)$.

We are not counting (p,q) as the same as (q,p). We are allowing $p = q$.

X/G is the "moduli space of line segments".

$X/G \cong [0,\infty)$ length

There is a line segment (p,q) and also (q,p), but $(p,q) \sim (q,p)$ so they have the same equivalence class: they give a single point in X/G. (no negative lengths)

~~What is the automorphism~~ Consider $X//G$. Objects are line segments & morphisms are like

$$(p,q) \xrightarrow{(g,p,q)} (p',q') \qquad \text{where } gp = p',\ gq = q',$$

as in the picture.

Given a groupoid C, we can form:

1) the set $\underline{C}$ of isomorphism classes of objects: $[x]$ where $[x]=[y]$ iff $x \cong y$.

2) for any $[x] \in \underline{C}$, a group $\mathrm{Aut}(x)$, where x is any representative of $[x]$. (Note: if $x \cong y$, then $\mathrm{Aut}(x) \cong \mathrm{Aut}(y)$ as groups.)

<u>Thm</u> Given a groupoid C, we can recover C from $\underline{C}$ and all the groups $\mathrm{Aut}(x)$ (one for each isomorphism class in $\underline{C}$).

Example: $C = \mathrm{FinSet}_0$

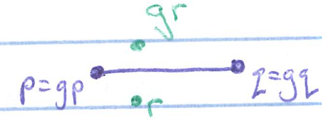
$\underline{C} \cong \mathbb{N}$ and for each $n \in \mathbb{N}$ we get a group S_n which we've seen is (isomorphic to) $\mathrm{Aut}(x)$ for any $x \in \mathrm{FinSet}_0$ with n elements.

Example: $C = X/\!/G$ where X is the set of line segments and G is the Euclidean group of the plane.

$\underline{C} \cong [0,\infty)$

In general $\underline{X/\!/G} \overset{=}{\cong} X/G$ because both are names for the set of equivalence classes $[x]$ where $x \sim y$ iff $y = gx$ $\exists g \in G$.

But $X/\!/G$ has more information, namely all the automorphism groups $\mathrm{Aut}(x)$, one for each equivalence class.



So: what's $\mathrm{Aut}((p,q))$?

$\mathrm{Aut}((p,q))$ ~~is the trivial group~~ if $p \neq q$: $\mathbb{Z}_2$.

If $p=q$: O_2 the group of all orthogonal 2×2 matrices, i.e. all rotations and reflections fixing $p \in \mathbb{R}^2$.