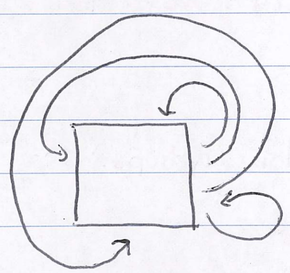
11/2/15

Groupoids

Def A morphism $f: x \to y$ in a category has an inverse $g: y \to x$ if $fg = 1_y$ & $gf = 1_x$.
If f has an inverse, it's unique so we write it as f^{-1}.
A morphism with an inverse is called an isomorphism.
If there's an isomorphism $f: x \to y$ we say x & y are isomorphic.

Def A groupoid is a category where all morphisms are isomorphisms.

Ex Any group G gives a groupoid with one object, $*$, and morphisms $g: * \to *$ corresponding to elements $g \in G$, with composition coming from multiplication in G.
Conversely any 1-object groupoid gives a group.
So "a group is a 1-object groupoid."
More generally, if C is any category & $x \in C$, the isomorphisms $f: x \to x$ form a group under composition, called the automorphism group $Aut(X)$.



$Aut(\square) \cong \mathbb{Z}_4$
(rotational symmetries)

Ex Given any category C there's a groupoid, the core of C, C_0, whose objects are those of C & whose morphisms are the isomorphisms of C, composed as before.

Ex If $FinSet = [\text{finite sets, functions}]$
then $FinSet_0 = [\text{finite sets, bijections}]$
and if n is your favorite n-element set, $Aut(n) = S_n$, the symmetric group
So $FinSet_0$ "unifies" all the symmetric groups

$\boxed{\text{Ex}}$ Suppose G is a group acting on a set X:

$$\alpha : G \times X \longrightarrow X$$
$$(g,x) \longmapsto gx$$

Often people form the set X/G, the quotient set where an element $[x]$ is an equivalence class of elements $x \in X$ where $x \sim y$ iff $y = gx$ for some $g \in G$.

But a "better" thing is to form the translation groupoid $X/\!/G$, where objects are elements $x \in X$
a morphism from x to y is a pair (g,x) where $g \in G$ and $gx = y$.

$$x \xrightarrow{(g,x)} y$$

the composite of

$$x \xrightarrow{(g,x)} y \quad \text{and} \quad y \xrightarrow{(h,y)} z$$

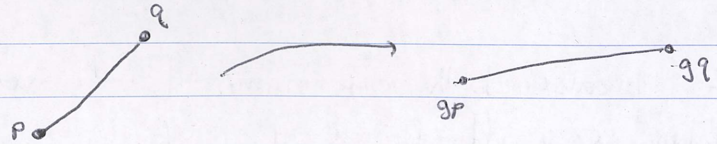
is

$$x \xrightarrow{(hg,x)} z$$

In X/G, we say x & y are equal if $gx = y$
In $X/\!/G$, we say they are isomorphic, or more precisely, we have a chosen isomorphism $(g,x) : x \longrightarrow y$.

To a first approximation, a "moduli space" is a set X/G, given some obvious topology, while a "moduli stack" is a group $X/\!/G$, where the sets of objects & morphisms have topologies.

$\boxed{\text{Ex}}$ Let X be the set of line segments in the Euclidean plane. Let G be the Euclidean group of the plane:
all bijections $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t. $|gp - gq| = |p-q|$
i.e. g preserves distances
G acts on X:



More precisely, $X = \mathbb{R}^2 \times \mathbb{R}^2$ & G acts on it via $g(p,q) = (gp, gq)$

We're not counting (p,q) as the same as (q,p).
We're allowing $p = q$.

(continued)

X/G is the "moduli space of line segments"
$X/G \cong [0,\infty)$

⌈There's a line segment (p,q) and also a line segment (q,p), but $(p,q)\sim(q,p)$. So they have some equivalence class: they give a single point in X/G⌋

Next consider $X/\!/G$

Now objects are line segments & morphisms are like:

$$(p,q) \xrightarrow{(g,p,q)} (p',q')$$

where $gp = p'$ & $gq = q'$, as in the picture.

Given a groupoid C, we can form:

1) the set $\underline{C}$ of isomorphism classes of objects:
   $[x]$ where $[x]=[y]$ iff $x \cong y$.
2) for any $[x] \in \underline{C}$, a group $\mathrm{Aut}(x)$, where x is any representative of $[x]$. (Note if $x\cong y$, then $\mathrm{Aut}(x) \cong \mathrm{Aut}(y)$ as groups.)

$\boxed{\text{Thm}}$ Given a groupoid C, we can recover C (up to equivalence) from $\underline{C}$ and all the groups $\mathrm{Aut}(x)$ (one for each isomorphism class in $\underline{C}$).

$\boxed{\text{Ex}}$ $C = \mathrm{FinSet}$

$\underline{C} \cong \mathbb{N}$

and for each $n \in \mathbb{N}$ we get a group S_n which we've seen is (iso. to) $\mathrm{Aut}(x)$ for any $x \in \mathrm{FinSet}$ with n elements.

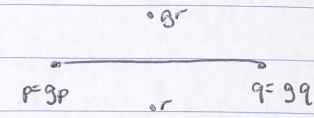
$\boxed{\text{Ex}}$ $C = X/\!/G$ where X is the set of line segments & G is the Euclidean group of the plane.

$\underline{C} \cong [0,\infty)$

In general, $\underline{X/\!/G} = X/G$ because both are names for the set of equivalence classes $[x]$ where $x\sim y$ iff $y = gx$ $\exists$ $g\in G$.
But $X/\!/G$ has more information, namely all the automorphism groups $\mathrm{Aut}(x)$, one for each equivalence class.

So in our example, what's $Aut((p,q))$?

It's $\mathbb{Z}_2$ if $p \neq q$; since there is a reflection preserving (p,q)



If $p=q$ it's group $O(2)$ of all orthogonal 2×2 matrices, i.e. all rotations & reflections fixing $p \in \mathbb{R}^2$.