Groupoids

Def. – A morphism $f: x \to y$ in a category has an <u>inverse</u> $g: y \to x$ if $fg = 1_y$ & $gf = 1_x$. If f has an inverse, it's unique, so we write it as f^{-1}. A morphism with an inverse is called an <u>isomorphism</u>. If there's an isomorphism $f: x \to y$, we say x & y are <u>isomorphic</u>.

Def. – A <u>groupoid</u> is a category where all morphisms are isomorphisms.

Example – Any group G gives a groupoid with one object, $*$, & morphisms $g: * \to *$ corresponding to elements $g \in G$, with composition coming from multiplication in G. Conversely, any 1-object groupoid gives a group. So a group is a 1-object groupoid. More generally, if C is any category & $x \in C$, the isomorphisms $f: x \to x$ form a group under composition, called the <u>automorphism group</u> $\mathrm{Aut}(X)$.

Example – Given any category C, there's a groupoid, the <u>core</u> C_0 of C, whose objects are those of C & whose morphisms are the isomorphisms of C, composed as before.

Example – If $\mathrm{FinSet} = [\text{finite sets, functions}]$, then $\mathrm{FinSet}_0 = [\text{finite sets, bijections}]$. And if n is your favorite n-element set, then $\mathrm{Aut}(n) = S_n$, the symmetric group. So FinSet_0 "unifies" all the symmetric groups.

Example – Suppose G is a group acting on a set X: $\alpha: G \times X \to X$, $(g,x) \mapsto gx$. Often people form the set X/G, the quotient set where an elt. $[x]$ is an equivalvence class of elts $x \in X$ where $x \sim y$ iff $y = gx$ for some $g \in G$. But a "better" thing is to from the <u>translation groupoid</u> $X/\!/G$, where: objects are elements $x \in X$, & a morphism from x to y is a pair (g,x) where $g \in G$ & $gx = y$; $x \xrightarrow{(g,x)} y$. The composite of $x \xrightarrow{(g,x)} y$ & $y \xrightarrow{(h,z)} z$ is $x \xrightarrow{(hg,x)} z$.

In X/G we say x & y are "equal" if $gx = y$; in $X/\!/G$ we say they are isomorphic, or more precisely, we have a chosen isomorphism $(g,x): x \to y$.

To a first approximation, a "moduli space" is a set X/G, given some obvious topology, while a "moduli stack" is a groupoid $X/\!/G$, where the set of objects & morphisms have topologies.

Example - Let X be the set of line segments in the Euclidean plane. Let G be the <u>Euclidean group</u> of the plane: all bijections $g: \mathbb{R}^2 \to \mathbb{R}^2$ that preserve distances. G acts on X:

More precisely, $X = \mathbb{R}^2 \times \mathbb{R}^2$ & G acts on it via $g(p,q) = (gp, gq)$.
We're not counting (p,q) the same as (q,p). We're allowing $p = q$.
X/G is the "moduli space of line segments". $X/G \cong [0,\infty)$.
Given a line segment (p,q), there's a line segment (q,p), but $(p,q) \sim (q,p)$, so they have the same equivalence class & hence give the same element of X/G.

Next consider $X/\!/G$. Now objects are line segments & morphisms are like: $(p,q) \xrightarrow{(g,(p,q))} (p',q')$ where $gp = p'$ & $gq = q'$ as in the picture.

Given a groupoid C, we can form:

1) the set $\underline{C}$ of isomorphism classes of objects: $[x]$ where $[x] = [y]$ iff $x \cong y$.

2) for any $[x] \in \underline{C}$, a group $\mathrm{Aut}(x)$ where x is any representative of $[x]$. Note if $x \cong y$ then $\mathrm{Aut}(x) \cong \mathrm{Aut}(y)$ as groups.

Thm. - Given a groupoid C, we can recover C up to equivalence from $\underline{C}$ & all the groups $\mathrm{Aut}(x)$, one for each isomorphism class in $\underline{C}$.

Example - $C = \mathrm{FinSet}_0$. $\underline{C} \cong \mathbb{N}$. & for each $n \in \mathbb{N}$ we get a group S_n which we've seen is isomorphic to $\mathrm{Aut}(x)$ for any $x \in \mathrm{FinSet}_0$ with n elements.

Example - $C = X/\!/G$ where X is the set of line segments & G is the Euclidean group of the plane. $\underline{C} \cong [0,\infty)$.

In general, $\underline{X/\!/G} = X/G$ because both are names for the set of equivalence classes $[x]$ where $x \sim y$ iff $y = gx$ for some $g \in G$.

But $X//G$ has more information, namely all the automorphism groups $\mathrm{Aut}(x)$, one for each equivalence class. So in our example, what's $\mathrm{Aut}((p,q))$? It's $\mathbb{Z}_2$ if $p \neq q$ since there's a reflection preserving (p,q). If $p=q$, it's the group $O(2)$ of all orthogonal 2×2 matrices, i.e. all rotations & reflections fixing $p \in \mathbb{R}^2$.