11/9/15

Moduli Spaces & Moduli Stacks

Given a groupoid C, let $\underline{C}$ be the set of isomorphism classes of objects. Often $\underline{C}$ will have the structure of a space (e.g. topological space, manifold, algebraic variety, scheme, ...). Then $\underline{C}$ is called a <u>moduli space</u>.

Example: if G is a group acting on a set X, we get a groupoid $X//G$, the <u>translation</u> groupoid, where:

- objects are elements of X
- morphisms are pairs: $x \in X$, $g \in G$, where $y = gx$.

$x \xrightarrow{(g,x)} y$

Then $\underline{X//G} \cong X/G$ where X/G has elements which are equivalence classes $[x]$ with $x \sim y$ when $y = gx$ for some $g \in G$.

← X weakly mod G.

Recall: Thm: The groupoid $X//G$ is equivalent to the groupoid with:

- one object $[x]$ for each $[x] \in X/G$
- one morphism $f : [x] \to [x]$ for each morphism $f : x \to x$, where x is any chosen representative of the equivalence class $[x]$.

If $[x] \neq [y]$ there are no morphisms between them.

We often call X/G a <u>moduli space</u>, and $X//G$ the <u>moduli stack</u>.

Last time we looked at an example:
"The moduli stack of line segments" in Euclidean geometry. Here

$$X = \mathbb{R}^2 \times \mathbb{R}^2 \ni (p,q)$$
$$G = O(2) \ltimes \mathbb{R}^2$$

Here G is the Euclidean group of the plane

We think of (p,q) as a line segment with a chosen first and second endpoint, which can be equal.

Then the moduli space is
$$X/G \cong [0,\infty)$$
$$[(p,q)] \mapsto |p-q|$$
the space of lengths.

The moduli stack $X/\!/G$ keeps track of symmetries:

$\mathrm{Aut}[(p,q)] \cong \mathrm{Aut}((p,q))$

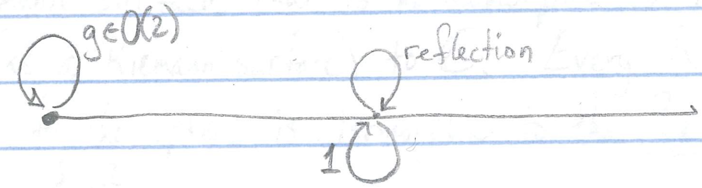
the subgroup G consisting of all $g \in G$ with $(gp, gq) = (p,q)$

$\mathrm{Aut}((p,q)) \cong \mathbb{Z}/2$ if $p \neq q$

$\mathrm{Aut}((p,q)) \cong O(2)$ if $p = q$

$\cong SO(2) \ltimes \underset{\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}}{\underset{\cup}{\mathbb{Z}_2}}$

So the moduli stack looks like:



Example: "The moduli stack of triangles"

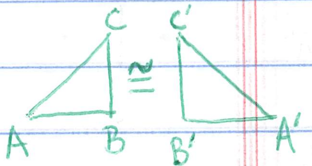
Let G be the Euclidean group as before, but now let X be the set of triangles: $X = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$.

These are triangles with named vertices that can be equal.

The moduli space X/G is the set of isomorphism classes of triangles. Now

$$X/G \cong [0,\infty)^3$$
$$[(p,q,r)] \longmapsto (|p-q|, |q-r|, |r-p|)$$

But does orientation matter?



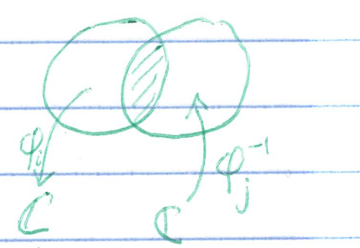
Here it seems if $p \neq q \neq r \neq p$, then (p,q,r) has as automorphisms only the identity.

If we define the triangles to be unordered triples of points in $\mathbb{R}^2$, an equilateral triangle would have S_3 as automorphisms, and isosceles would have $S_2 = \mathbb{Z}/2$. This gives a more interesting moduli stack.

Example: A Riemann surface is a 2-dimensional smooth manifold with charts

$$\varphi_i : U_i \longrightarrow \mathbb{C}$$

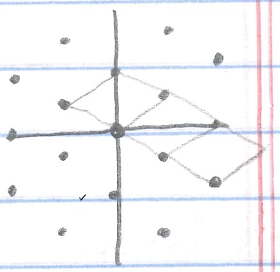
such that $\varphi_i \varphi_j^{-1}$ is analytic (=holomorphic)



Every Riemann surface that is homeomorphic to the plane is isomorphic (as a Riemann surface) to $\mathbb{C}$. Every Riemann surface homeomorphic to the sphere is isomorphic to the Riemann sphere $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$.

There are lots of non-isomorphic ways to make a torus into a Riemann surface — these are elliptic curves.

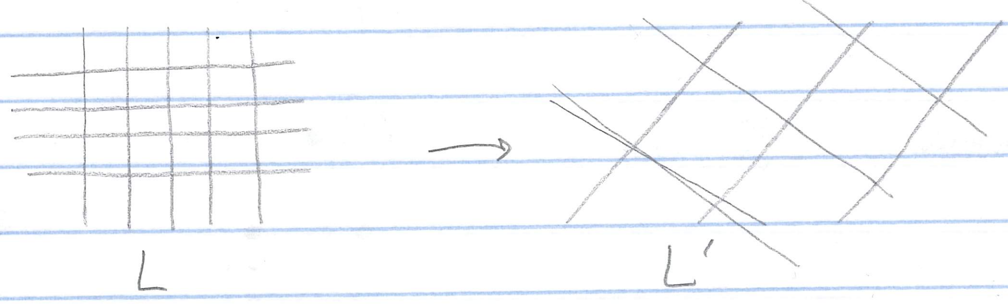
Every elliptic curve is isomorphic to one of this form: take a *lattice* $L \subseteq \mathbb{C}$, i.e. a subgroup of $(\mathbb{C}, +, 0)$ that's isomorphic to $\mathbb{Z}^2$, and form $\mathbb{C}/L$, getting a torus with obvious charts $\varphi_i : U_i \to \mathbb{C}$, and thus an elliptic curve.



When do two lattices L & L' give isomorphic elliptic curves: $\mathbb{C}/L \cong \mathbb{C}/L'$?

Answer: when $L' = \alpha L$ for some nonzero $\alpha \in \mathbb{C}$.

(the shape of the parallelograms are the same, but the size and rotation may be different.)



There's a groupoid $\underline{C}$ with

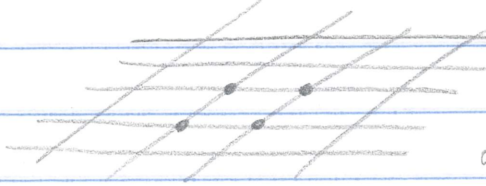
- elliptic curves as objects
- isomorphisms of Riemann surfaces as morphisms

and we're seeing $\underline{C} \cong X/G$ where X is the set of lattices and $G = \mathbb{C}^*$ (nonzero complex numbers with multiplication).

So X/G is called the *moduli space of elliptic curves*, and $X//G$ is the *moduli stack* of elliptic curves.

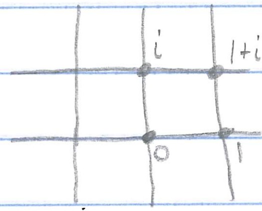
There are two elliptic curves with a bigger automorphism group:

Typical lattice/ elliptic curve



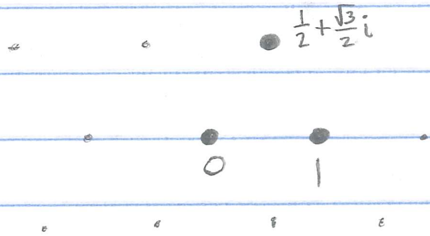
has $\mathbb{Z}/2$ as symmetries from $180°$ rotation $(-1)^2=1$

Gaussian elliptic curve



has $\mathbb{Z}/4$ as automorphisms: $i^4=1$

Eisenstein elliptic curve



has $\mathbb{Z}/6$ as automorphisms