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Klein Geometry

Def: A <u>homogeneous</u> <u>G-space</u> for some group G is a <u>G-set</u> X, i.e. a set X with a map

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto gx$$

Such that $g_1(g_2 x) = (g_1 g_2)(x)$, $1x = x$
which is transitive, i.e.

$$\forall x,y \in X \quad \exists g \in G, \quad gx = y$$

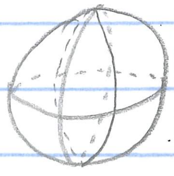
Example: In Euclidean plane geometry, $G = O(2) \ltimes \mathbb{R}^2$ is the <u>Euclidean group</u> and $X = \mathbb{R}^2$ is the <u>Euclidean plane</u> with $g = (r,t) \in O(2) \ltimes \mathbb{R}^2$ acting on $x \in \mathbb{R}^2$ by: $gx = rx + t$.

In non-Euclidean geometry, the parallel postulate fails. Here, the Euclidean group is replaced by some other 3-dimensional Lie group

Example: In spherical geometry, $G = O(3)$, i.e.

$$G = \{ g : \mathbb{R}^3 \to \mathbb{R}^3 : g \text{ is linear } \& \ gx \cdot gy = x \cdot y \text{ for all } x, y \in \mathbb{R}^3 \}$$

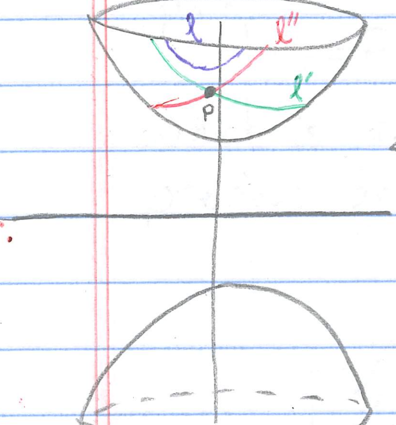
$$X = S^2 = \{ x \in \mathbb{R}^3 : x \cdot x = 1 \}$$



Here we can define a set of <u>lines</u>, namely great circles, but any two distinct lines intersect in two points, so parallel postulate fails, though other axioms of Euclid hold.

Example: In hyperbolic geometry we let $\mathbb{R}^{2,1}$ be $\mathbb{R}^3$ with the dot product $(x,y,z)\cdot(x',y',z') = xx' + yy' - zz'$ and let $G = O(2,1) = \{ g : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1} : g \text{ is linear } \& \ gx \cdot gy = x \cdot y \ \ \forall x, y \in \mathbb{R}^{2,1} \}$

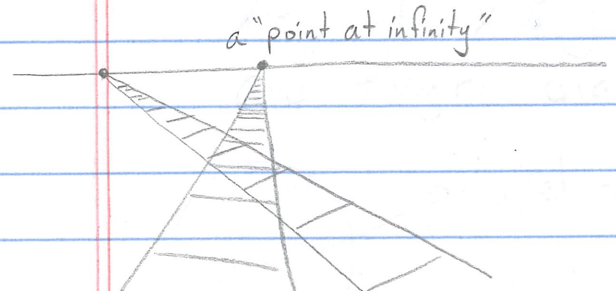
and $X = H^2 = \{ x \in \mathbb{R}^{2,1} : x \cdot x = -1 \}$
$= \{ (x,y,z) : x^2 + y^2 - z^2 = -1 \}$
the hyperboloid.

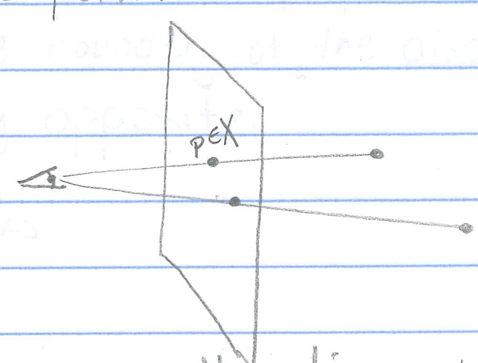


As in spherical geometry, we can define a <u>line</u> to be an intersection of X with some plane through the origin.

The parallel postulate fails because for any line ℓ & any point p not on that line, there are infinitely many lines (e.g ℓ' and ℓ'') containing p but parallel to ℓ (i.e. not intersecting ℓ).

In projective (plane) geometry every pair of distinct lines intersects in exactly one point!





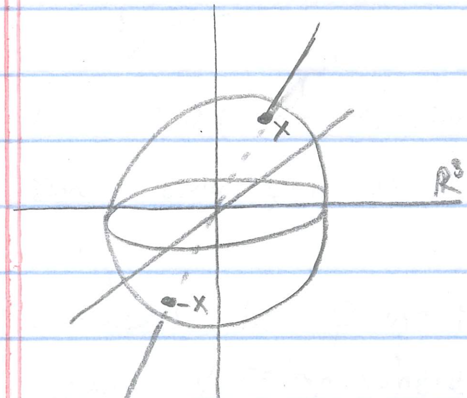
Points in projective geometry are really lines through the origin (your eye)

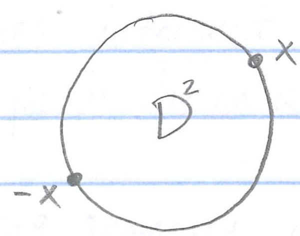
In projective geometry $X = \mathbb{R}P^2 = \{$lines through the origin in $\mathbb{R}^3\}$

and $G = GL(3,\mathbb{R}) = \{g : \mathbb{R}^3 \to \mathbb{R}^3 : g$ linear, invertible$\}$
... or, since transformations $x \mapsto \alpha x$ $(0 \neq \alpha \in \mathbb{R})$
act trivially on X, we can use the <u>projective general linear group</u>

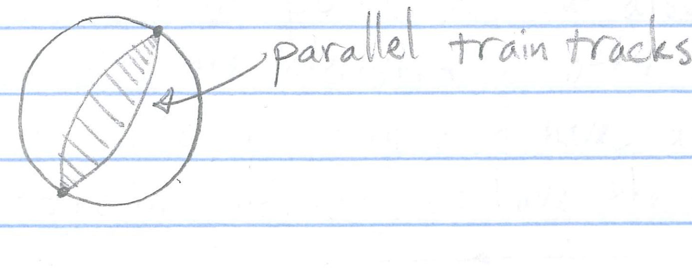
$$G = PGL(3,\mathbb{R}) = GL(3,\mathbb{R}) / \{\alpha I : \alpha \in \mathbb{R}\}$$

which is an 8-dimensional group, as opposed to the previous groups, which were 3-dimensional.

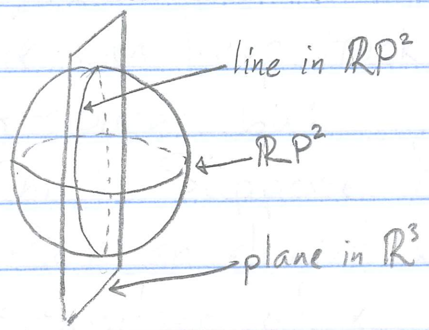




$\mathbb{R}P^2$ can be identified with $S^2/\sim$ where
$x \sim y$ iff $y = \pm x$,
or therefore with $D^2/\sim$ where
$x \sim y$ if x & y are the boundary of the disc D^2 and they're diametrically opposite.



So $\mathbb{R}P^2$ can be seen as $\mathbb{R}^2$ (homeomorphic to the interior of D^2) together with "points at infinity" coming from the boundary of the disc.
We can define a <u>line</u> in $\mathbb{R}P^2$ to be a plane through the origin in $\mathbb{R}^3$, which contains lots of points in $\mathbb{R}P^2$ (which were lines through the origin).



Any pair of distinct lines intersect in a unique point, and any pair of distinct points lie on a unique line.

Indeed, in projective plane geometry, any theorem has a "dual" version where the role of points and lines are switched. This is a special case of duality for posets, with $p < l$ meaning p lies on l.

Klein noticed that in all the kinds of geometry mentioned so far, we actually have two homogeneous G-space: the set of points X, but also the set of lines Y.
We also are interested in other homogeneous G-spaces, e.g. in 3d geometry we'd have a set of planes, or in 2d projective geometry we have the set of flags:

i.e. point-line pairs, where the point lies on the line, etc.

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So Klein's idea was:
a geometry is simply a group, and a type of figure (point, line, flag, triangle, etc.) is a homogeneous G-space X, with an element $x \in X$ being a figure of that type.

We can keep track of (classify) all these homogeneous G-spaces by:

Thm: Suppose X is a homogeneous G-space. Pick an element $x \in X$ and let $H \subseteq G$ be the stabilizer of x, i.e. the subgroup

$$H = \{g \in G : gx = x\}.$$

Then let G/H be the set of equivalence classes $[g]$ where $g \sim g' \Leftrightarrow g' = gh$. Then G/H is a homogeneous G-space with: $g[g'] = [gg']$

and as G-spaces we have

$$X \cong G/H \quad \text{via } \alpha : gx \mapsto [g] \quad \forall g \in G$$

with the obvious inverse.

Here α is a map of G-spaces: $\alpha(gx) = g\alpha(x)$.

This allowed Klein to redefine a type of figure to be simply a subgroup of G, since a subgroup $H \subseteq G$ gives a transitive G-space G/H, and every transitive G-space is isomorphic to one of those.