Moduli Spaces & Moduli Stacks

Given a groupoid C, let $\underline{C}$ be the set of isomorphism classes of objects. Often $\underline{C}$ will have the structure of a space (e.g. a topological space, a manifold, an algebraic variety, a scheme, ...). Then $\underline{C}$ is called a <u>moduli space</u>.

[Ex] If G is a group acting on a set X, we get a groupoid $X/\!/G$, the <u>translation</u> groupoid, where:

- objects are elements of X
- morphisms $x \xrightarrow{(g,x)} y$ are pairs $x \in X$, $g \in G$, where $y = gx$.

Then $\underline{X/\!/G} \cong X/G$ where X/G has elements $[x]$ with $x \sim y$ when $y = gx$ for some $g \in G$.

Recall

[Thm] The groupoid $X/\!/G$ is equivalent to the groupoid with:

- one object $[x]$ for each $[x] \in X/G$
- one morphism $f: [x] \longrightarrow [x]$ for each morphism $f: x \longrightarrow x$ where x is any chosen representative of the equivalence class $[x]$.

If $[x] \neq [y]$ there are no morphisms between them.

We often call X/G a <u>moduli space</u>, and $X/\!/G$ the <u>moduli stack</u>.

Last time we looked at an example:

[Ex] "The moduli stack of line segments" in Euclidean geometry.

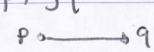
Here $X = \mathbb{R}^2 \times \mathbb{R}^2 \ni (p,q)$, $G = O(2) \ltimes \mathbb{R}^2$

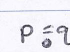
Here G is the Euclidean group of the plane and we think of (p,q) as a line segment with a chosen 1st & 2nd endpoint, which can be equal.

Then the moduli space is $X/G \cong [0,\infty)$ the space of lengths.

$$[(p,q)] \mapsto |p-q|$$

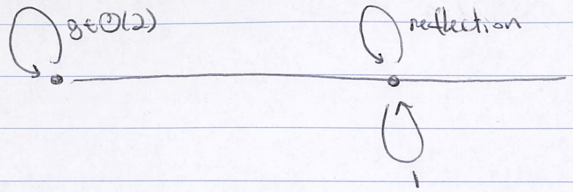
The moduli stack $X/\!/G$ keeps track of symmetries: $\mathrm{Aut}[(p,q)] \cong \mathrm{Aut}((p,q))$ is the subgroup of G consisting of all $g \in G$ with $(gp, gq) = (p,q)$

$\mathrm{Aut}((p,q)) \cong \mathbb{Z}/2$ if $p \neq q$ 

$\mathrm{Aut}((p,q)) \cong O(2)$ if $p = q$ 

$\cong SO(2) \ltimes \mathbb{Z}_2$

So the moduli stack looks like



Ex "The moduli stack of triangles"

Let G = the Euclidean group as before, but now let X be the set of triangles:

$X = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$

These are triangles with named vertices that can be equal.

The moduli space X/G is the set of isomorphism classes of triangles.

Now $X/G \cong [0,\infty)^3$

$[(p,q,r)] \mapsto (|p-q|, |q-r|, |r-p|)$

Here it seems that if p,q,r are all distinct, (p,q,r) has as automorphisms only the identity.

If we define a triangle to be an unordered triple of points in $\mathbb{R}^2$, an equilateral triangle would have S_3 as automorphisms, and isoceles would have $S_2 = \mathbb{Z}/2$.

This gives a more interesting moduli stack.

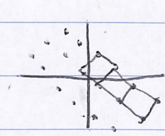
Ex A Riemann surface is a 2-dim. smooth manifold with charts $\varphi_i : U_i \to \mathbb{C}$ s.t. $\varphi_i \varphi_j^{-1}$ is analytic (=holomorphic)

- Every Riemann surface that's homeomorphic to the plane is isomorphic (as a Riemann surface) to $\mathbb{C}$.
- Every Riemann surface homeomorphic to the sphere is isomorphic to the <u>Riemann sphere</u> $\mathbb{C}P^1 \cong \mathbb{C} \cup \{0\}$

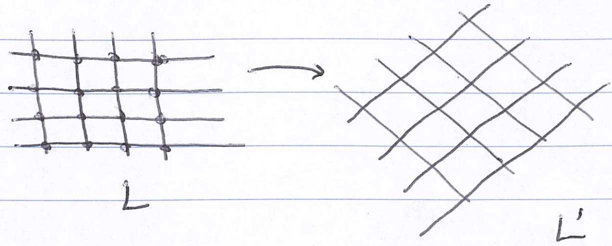
There are lots of nonisomorphic ways to make a torus into a Riemann surface — these are <u>elliptic curves</u>.

Every elliptic curve is isomorphic to one of this form:

take a <u>lattice</u> $L \subseteq \mathbb{C}$, i.e. a subgroup of $(\mathbb{C}, +, 0)$ that's isomorphic to $\mathbb{Z}^2$, and form $\mathbb{C}/L$, getting a torus with obvious charts $\varphi_i : U_i \to \mathbb{C}$, and thus an elliptic curve.



When do 2 lattices L & L' give isomorphic elliptic curves: $\mathbb{C}/L \cong \mathbb{C}/L'$?
Answer: iff $L' = \alpha L$ for some nonzero $\alpha \in \mathbb{C}$

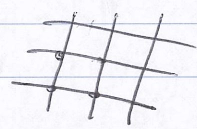


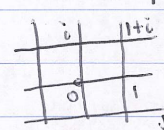
There's a groupoid $\underline{C}$ with
- elliptic curves as objects
- isomorphisms of Riemann surfaces as morphisms

and we're seeing
$\underline{C} \simeq X/\!/G$
where X is the set of lattices & $G = \mathbb{C}^*$ (nonzero complex numbers with multiplication)

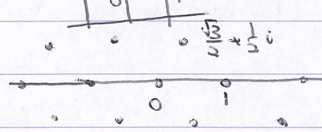
So X/G is called the <u>moduli space of elliptic curves</u>, and $X/\!/G$ is the <u>moduli stack</u> of elliptic curves.

There are 2 elliptic curves with a bigger automorphism group:

typical elliptic curve  has $\mathbb{Z}/2$ as symmetries - 180° rotation

Gaussian elliptic curve 

has $\mathbb{Z}/4$ as automorphisms $i^4 = 1$

Eisenstein elliptic curve 

has $\mathbb{Z}/6$ as automorphisms