11/30/15

Enriched Categories & internal monoids

A monoid is "the same" as a 1-object category: if you have a category C with one object x, there's a monoid $\hom(x,x)$ with multiplication $\circ : \hom(x,x) \times \hom(x,x) \longrightarrow \hom(x,x)$
Conversely given a monoid M you can build a category with one object x and $\hom(x,x) = M$, with composition being multiplication in M.

More generally suppose V is a monoidal category, i.e. a category with a tensor product: $\otimes : V \times V \longrightarrow V$.
obeying some rules.

Then recall a V-enriched category C has a class of objects and for any objects $x, y \in C$, a "hom-object" $\hom(x,y) \in V$ & composition morphisms:
$\circ : \hom(x,y) \otimes \hom(y,z) \longrightarrow \hom(x,z)$

A 1-object V-enriched category is the same as a monoid internal to V, or monoid in V, i.e. an object $M \in V$ with a multiplication
$m : M \otimes M \longrightarrow M$ that's associative and unital.

Ex Suppose $V = \mathrm{AbGp}$ with the usual tensor product of abelian groups. Then a monoid in V is called a ring. (ring has unit, but rng does not)
It's an abelian group M, with a multiplication $m : M \otimes M \longrightarrow M$ an abelian group homomorphism, i.e. a function $m : M \times M \longrightarrow M$ that's linear in each argument:
$(a+b) \cdot c = a \cdot c + b \cdot c$
$a \cdot (b + c = a \cdot b + a \cdot c$

Ex If $V = R\mathrm{Mod}$ for some comm. ring R, a monoid in V is called an R-algebra.

Ex If $V = \mathrm{Top}$ with usual product $\times$ of top'l spaces as $\otimes$, a monoid in V is called a topological monoid.

Back to our favorite example: Klein geometry

Let G be a group, and let GRel be the category with

G-sets as objects

G-invariant relations as morphisms.

This a CABA-enriched category. So if we take one object, i.e. one G-set X, we can form a 1-object CABA-enriched category with X as the only object $\hom(X,X)$ is the only homset, or "hom-CABA".

$\boxed{\text{Ex}}$ Projective plane geometry

Take $G = PGL(3, \mathbb{R})$

$Y = \{\text{flags}\}$

$= \{(p, L) : p \subseteq \mathbb{R}^3$ is a 1-dim'l subspace,

$L \subseteq \mathbb{R}^3$ is a 2-dim'l subspace,

$p \subseteq L\}$

$\hom(Y,Y)$ is a monoid in CABA. What is it like?

Instead of describing all the elements, let's just describe the atoms.

In general, given any group G and any G-sets X, Y, what are the atoms in $\hom(X,Y)$ like?

They're invariant relations $R : X \to\!\!\!\to Y$

i.e. $R \subseteq X \times Y$ s.t. $(x,y) \in R \Rightarrow (gx, gy) \in R$

But they are the smallest nonempty subsets of this form. So, any atom R must contain a point (x,y), and thus all points of the form (gx, gy) with $g \in G$. Indeed, any orbit $\{(gx, gy) : g \in G\} \subseteq X \times Y$ is an atom in $\hom(X,Y)$.

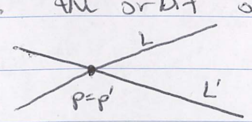
So, if

$G = PGL(3, \mathbb{R})$

$Y = \{\text{flags}\}$

the atoms in $\hom(Y,Y)$ are the orbits of G acting on $Y \times Y$.

e.g. the orbit of this

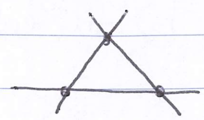


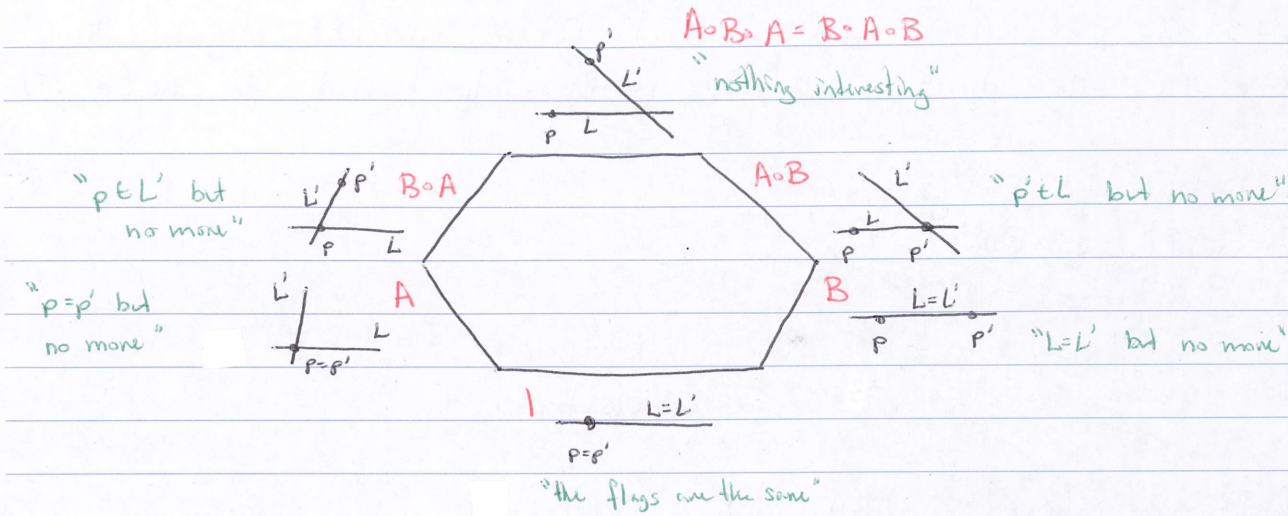
$x = (p, L)$

$y = (p', L')$

is the set of all pairs of flags sharing the point (and no more!).

Last time we saw all 6 atoms in hom(Υ,Υ):





The identity $1 \in \hom(\Upsilon,\Upsilon)$ is "two flags are the same"

Note we can compose invariant relation & $1 \circ 1 = 1$

Let $A \in \hom(\Upsilon,\Upsilon)$ be "having the same point but no more"

$A \circ A = A \cup 1$

If you change the line on a flag twice, the result could be changing the line or getting back the original flag.

Let $B \in \hom(\Upsilon,\Upsilon)$ be "having the same line but no more" and "changing the point"

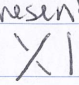
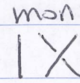
$B \circ B = B \cup 1$

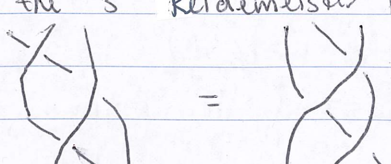
$A \circ B$ is one of our atoms, "$p' \in L$ but no more"

$B \circ A$ is another atom, "$p \in L'$ but no more"

$A \circ B \circ A = B \circ A \circ B$ is "nothing interesting"

In fact this is a presentation for our monoid in CABA, $\hom(\Upsilon,\Upsilon)$.

If we draw A as  and B as  then $A \circ B \circ A = B \circ A \circ B$ is called the "3rd Reidemeister move" or "Yang-Baxter equation":



This is the only relation in B_3, the 3-strand braid group.