LINEAR ALGEBRAIC GROUPS: LECTURE 1

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1. Introduction

Recall that $GL(n,k)$ is the collection of linear, invertible transformations on k^n, where k is a field.

Definition (rough)**.** A <u>linear algebraic group</u> G is a subgroup of $GL(n,k)$ satisfying

$$G \;=\; \{g \in GL(n,,k) : P_1(g) = \cdots = P_N(g) = 0\}\,,$$

where $P_i : M(n,k) \to k$ are polynomials in matrix entries.

Note: We only require a finite number of polynomials to define this group, as the ring is Noetherian and satisfies the ascending chain condition.

Example 1. Recall that the <u>general linear group</u> is

$$GL(n,k) \;=\; \{g \in M(n,k) : \det g \neq 0\}\,.$$

$GL(n,k)$ is "the king" of linear algebraic groups.

Example 2. We define the <u>special linear group</u> as

$$SL(n,k) \;=\; \{g \in GL(n,k) : \det g = 1\}\,.$$

This is an algebraic group, as determinant is a polynomial in matrix entries. This group reduces the size of the center in comprison to $GL(n)$. Note that the center of $SL(n)$ is all matrices of the form

$$\begin{pmatrix} \alpha & 0 & \cdots & & 0 \\ 0 & \alpha & & & \\ \vdots & & \ddots & & \vdots \\ & & & \alpha & 0 \\ 0 & & \cdots & 0 & \alpha \end{pmatrix},$$

where α is an n^{th} root of unity.

Example 3. We define the <u>orthogonal group</u> as

$$\begin{aligned} O(n,k) \;&=\; \left\{g \in GL(n,k) : g^T g = 1\right\}. \\ &=\; \{g \in GL(n,k) : gv \cdot gw = v \cdot w \text{ for all } v,w \in k^n\}\,. \end{aligned}$$

This is the usual dot product,

$$v \cdot w \;=\; \sum_{i=1}^{n} v_i w_i.$$

Since $g^T g = 1$, we know that $\det g = \pm 1$, or $(\det g)^2 = 1$. Hence, $O(n)$ is defined by n^2 quadratic equations. Moreover,

$$gv \cdot gw \;=\; vg^T \cdot gw \;=\; v \cdot w$$

if and only if $g^T g = 1$, os our definitions are equivalent. Conceptually, $O(n)$ is the collection of all reflections and rotations, as well as the composition of those transformations.

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Example 4. Going a step further, we have the special orthogonal group

$$\begin{aligned} SO(n,k) &= \left\{g \in GL(n) : g^T g = 1 \text{ and } \det g = 1\right\} \\ &= \{g \in O(n) \cap SL(n)\}. \end{aligned}$$

Note that the intersection of two algebraic (sub)groups is an algebraic group. This particular group, $SO(n)$, can be thought of as the collection of rotations in k^n.

Example 5. The Euclidean group, $E(n,k)$ can be thought of as all maps $f : k^n \to k^n$ of the form

$$f(x) = Rx + v, \qquad \text{where} \qquad R \in SO(n,k) \text{ and } v \in k^n.$$

For $n = 2$ and $k = \mathbb{R}$, this is the symmetry group of the Euclidean plane. These are the transformations which preserve lines, distances, and angles and orientation. (*Note: If we choose $O(n)$ instead of $SO(n)$, we would not preserve orientation*).

At a glance, it doesn't appear as if this is a subgroup of $GL(n)$. First of all, it is indeed a group. Composition of (R,v) with (R',v') on any $x \in k^n$ yields

$$R(R'x + v') + v = RR'x + Rv' + v,$$

and $RR' \in SO(n)$ while $Rv' + v \in k^n$. Inverses are also clear. For (R,v), define

$$(R,v)^{-1} := (R^{-1}, -R^{-1}v).$$

Finally, the identity is the element $(1,0)$. This is certainly a group, but is it an algebraic group? We can write our transformations as

$$\phi : (R,v) \mapsto \left(\begin{array}{ccc|c} & R & & v \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) \in GL(n+1,k).$$

Note that

$$\phi(R,v) \circ \phi(R',v') = \left(\begin{array}{ccc|c} & RR' & & \begin{matrix} Rv' \\ + \\ v \end{matrix} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) = \phi\left((R,v) \circ (R',v')\right),$$

so we have a group isomorphism, and we can consider

$$E(n) = \left\{ g \in GL(n+1) : g = \left(\begin{array}{ccc|c} & R & & v \\ \hline 0 & \cdots & 0 & 1 \end{array}\right), \text{ for } R \in SL(n) \text{ and } v \in k^n \right\}.$$

2. A Curious Postulate

Most of the Euclid's postulate have been accepted since the publication of *The Elements*. However, many mathematicians sought to show that postulate 5, the parallel postulate, follows from the others. Here's a rephrased equivalent version of the postulate, attributed to Scottish mathematician John Playfair.

(Almost) Playfair's Axiom. *In a plane, given a line and a point not on it, exactly one line parallel to the given line can be drawn through the point.*

Playfair's original version claims there is "at most" one line parallel, but the above is closer to the interpretation of Euclid's intent. To make this a bit more symbolic, let L be the collection of lines in the plane, and let P be the collection of points. Let I be the operator "is incident to". Then the postulate/axiom can then be written as

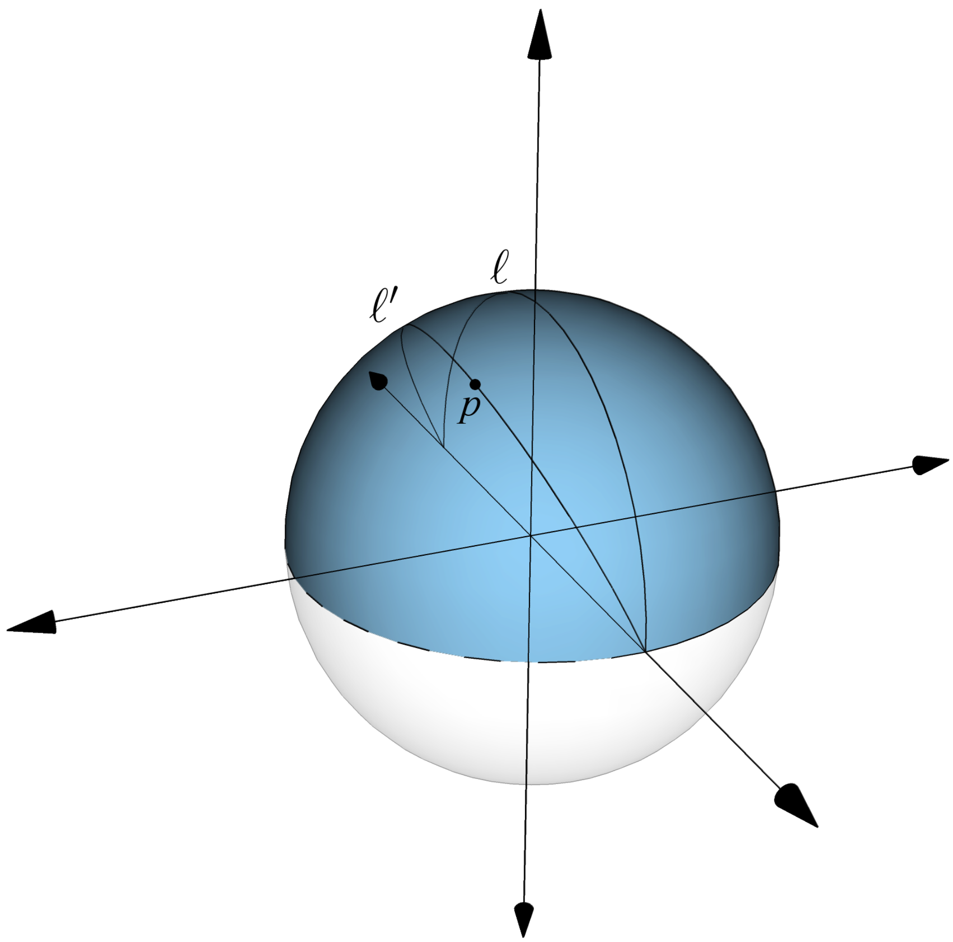
$$\forall p \in P, \forall \ell \in L : \; p\mathrm{I}\ell \Rightarrow \forall p'(\neg p'\mathrm{I}\ell), \; \exists! \ell' \in L \; (p'\mathrm{I}\ell' \wedge \neg\exists q(q\mathrm{I}\ell \wedge \neg q\mathrm{I}\ell')).$$

As it turns out, not only is the parallel postulate not provable from Euclid's other postulates, abandoning it leads to alternative, non-Euclidean geometry. Let's take a short look at two such model planes.

Elliptic Geometry. Start with the space S^2, the unit sphere satisfying $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ Construct the real projective space as a quotient,

$$\mathbb{R}P^2 \;=\; S^2/\sim,$$

where $p \sim -p$ (i.e., we identify antipodal points). We then let P be any points in $\mathbb{R}P^2$, and declare L to be the collection of great circles. In this case, given a line ℓ (a great circle) and any point p not in that line , any line ℓ' through p (again, a great circle) will intersect ℓ in exactly one point.

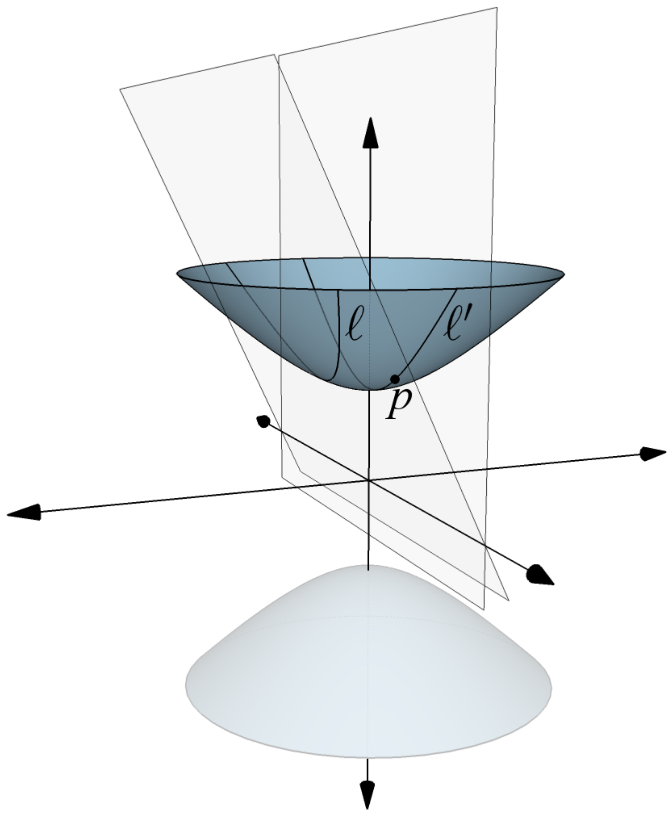


Thus the claimed nonintersecting line ℓ' fails to exist.

Hyperbolic Geometry. In this case, we begin with H_2, the hyperboloid of two sheets satisfying $x^2+y^2-z^2+1=0$. We again construct a quotient,

$$H \;=\; H_2/\sim,$$

where $p \sim -p$. This leaves us with the top "bowl". Now, we let P be the collection of points on this bowl, and let L be the collection of curves defined by the intersection of this bowl with planes throught the origin. In this case, it is easy to find an ℓ' which fails to intersect an original ℓ through a point p' not in ℓ.



However, there is more than enough wiggle room to allow many such elements in L to fit without intersecting our original ℓ. Thus, the uniqueness of ℓ' stipulated in the parallel postulate fails to be satisfied.