LINEAR ALGEBRAIC GROUPS: LECTURE 10

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1. Groups Within Categories

Last lecture, the duality between goemetry and commutative algebra allowed us to look at geometric spaces through their algebras of regular functions on those spaces. We claimed that

$$\begin{tikzcd}
 & \text{spaces} & & \text{algebras} \\
\mathrm{AlgSet}_k \arrow[r, "\sim"] & \mathrm{AffVar}_k \arrow[d, hook] & = & \mathrm{AffAlg}_k^{\mathrm{op}} \arrow[d, hook] \\
 & \mathrm{AffSch}_k & := & \mathrm{CommAlg}_k
\end{tikzcd}$$

The functor $\mathrm{AlgSet}_k \xrightarrow{\sim} \mathrm{AffAlg}_k^{\mathrm{op}}$ sends any algebraic set $S \subseteq X$ (a subset of a finite dimensional vector space, picked out as the zeros of a finite set of polynomials) to the algebra $k[S]$ consisting of functions $f : S \to k$ such that $f = P\big|_S$ for some $P \in k[X]$, the restriction of a polynomial on X to S. We saw that $k[S]$ is an affine algebra - finitely generated and lacking nonzero nilpotents, under the assumption that k is algebraically closed. In fact, <u>any</u> affine algebra is isomorphic to $k[S]$ for some $S \in \mathrm{AlgSet}_k$, which is unique up to isomorphism.

A morphism of algebraic sets $S \subseteq X$ and $S' \subseteq X'$.

$$\varphi : S \to S'$$

is a regular function from X to X' which sends S into S'. Our functor sends this φ to the algebra homomorphism

$$\varphi^* : k[S'] \to k[S]$$

given by

$$\varphi^*(f)(x) \ := \ f\left(\varphi(x)\right),$$

where $f \in k[S'],\ x \in S$ and $\varphi(x) \in S'$. In fact, every algebra homomorphism $\alpha : k[S'] \to k[S]$ equals φ^* for a unique morphism of algebraic sets $\varphi : S \to S'$.

We now wish to define groups <u>in</u> $\mathrm{AffVar}_k \cong \mathrm{AlgSet}_k \subseteq \mathrm{AffSch}_k$. To do this, we require these three categories to have finite products, i.e. binary products (denoted $'\times'$) and a terminal object (denoted 1).

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Definition. If $\mathcal{C}$ is a category with finite products, a group G in $\mathcal{C}$, or a group G internal to $\mathcal{C}$, or a group object G in $\mathcal{C}$ is an object $G \in \mathcal{C}$ with a multiplication morphism

$$M : G \times G \to G,$$

an identity assigning morphism

$$\mathrm{id} : 1 \to G,$$

and an inverse morphism

$$\mathrm{inv} : G \to G.$$

These must obey associativity,

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times 1_G"] \arrow[d, "1_G \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

left and right union laws,

$$\begin{tikzcd}
1 \times G \arrow[d, "\mathrm{id} \times 1_G"] & G \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "1_G"] & G \times 1 \arrow[d, "1_G \times \mathrm{id}"] \\
G \times G \arrow[r, "m"] & G & G \times G \arrow[l, "m"']
\end{tikzcd}$$

and left and right inverse laws,

$$\begin{tikzcd}
G \times G \arrow[dd, "1_G \times \mathrm{inv}"] & G \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "!"] & G \times G \arrow[dd, "\mathrm{inv} \times 1_G"] \\
 & 1 \arrow[d, "\mathrm{id}"] & \\
G \times G \arrow[r, "m"] & G & G \times G \arrow[l, "m"']
\end{tikzcd}$$

where ! is the unique morphism to the terminal object and Δ is the diagonal or duplication map.

Of course, this requires finite products.

Theorem. $\mathrm{AffSch}_k \cong \mathrm{CommAlg}_k^{\mathrm{op}}$ *and its subset* $\mathrm{AffVar}_k \cong \mathrm{AffAlg}_k^{\mathrm{op}}$ *have finite products.*

Proof. To prove AffSch_k has finite products, we can prove that $\mathrm{CommAlg}_k$ has finite coproducts. To do this, we will prove it has an initial object (an object with one morphism to any object). Our initial object, a commutative algebra over k, is the field k itself. Given any $A \in \mathrm{CommAlg}_k$, there exists a unique homomorphism from k into A, namely $!(1_k) = 1_A$. Then for any $c \in k$,

$$!(c \cdot 1_k) \;=\; c \cdot 1_A \;=\; c1_A \in A.$$

We can view k as the set of regular valued functions on a single point, or if you prefer, "k-valued fucntions on a single point".

The coproduct of two commutative algebras A and B is the sum $A + B \in \mathrm{CommAlg}_k$ satisfying

$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "g"] \\ & A+B \arrow[d, "\exists!"] & \\ & Q & \end{tikzcd}$$

Note that our sum, $'+'$, is not direct sum as there is no homomorphism $A \to A \oplus B$. If so, we would have maps

$$\begin{array}{rccc} i: & A & \to & A \oplus B \\ & a & \mapsto & (a,0) \end{array} \qquad \begin{array}{rccc} j: & B & \to & A \oplus B \\ & b & \mapsto & (0,b). \end{array}$$

These do not preserve the multiplicative identity, as

$$i(1_A) \;=\; (1,0) \;\neq\; (1,1) \;=\; 1_{A\oplus B}.$$

Instead, our coproduct is the tensor product $A \otimes B = A \otimes_k B$. This is the usual tensor product of vector spaces, with multiplication defined as

$$(a \otimes b)(a' \otimes b') \;=\; aa' \otimes bb',$$

having the multiplicative identity element $1_A \otimes 1_B$, and setting

$$i_A(a) \;:=\; a \otimes 1_B, \qquad i_B(b) \;:= 1_A \otimes b.$$

Our definition of i_A and i_B preserve both the multiplicative identity and inverses, so they are actually homomorphisms. Moreover, our commutative diagram becomes

$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "g"] \\ & A\otimes B \arrow[d, "!h"] & \\ & Q & \end{tikzcd}$$

Given such homomorphisms f and g into a commutative algebra Q, we can then define

$$h(a \otimes b) \;:=\; f(a)g(b),$$

to preserve the commutativity of the diagram. In this manner, the product of two spaces X and Y in the world of geometry translates to the algebra of functions on $X \times Y$ as $A \otimes B$, where A is the algebra of regular functions on X, and B is the algebra of regular functions on Y.

Finally, the subset $\mathrm{AffAlg}_k \subseteq \mathrm{CommAlg}_k$ has the same initial object k and binary product $\otimes$. Under the assumption that k is finitely generated without nilpotents, when A and B are finitely generated without nilpotents, so is their binary product $A \otimes B$. $\square$

2. The Place of Linear Algebraic Groups

Now we can study algebraic groups in a 20th century style.

Definition. A Group in $\mathrm{AffSch}_k \cong \mathrm{CommAlg}_k^{\mathrm{op}}$ is called an affine group scheme. A group in $\mathrm{AffVar}_k \cong \mathrm{AffAlg}_k^{\mathrm{op}}$ is called an affine algebraic group.

But there's a problem for us - this is a course on linear algebraic groups! Where do they actually fit?

Theorem. *Any linear algebraic group gives an affine algebraic group, and conversely any affine algebraic group gives a linear algebraic group.*

There is an equivalence of categories here.

Proof. (Sketch) A linear algebraic group is a subgroup

$$G \;=\; \{g \in GL(n,k) : P_1(g) = \cdots = P_n(g) = 0\}\,,$$

where $P_i : GL(n,k) \to k$ are polynomials, i.e. regular functions. How can we see G as an object in affine varieties, $\mathrm{AffVar}_k \cong \mathrm{AffAlg}_k^{\mathrm{op}}$? Our $G \subseteq GL(n,k)$ consists of matrices satisfying a bunch of polynomials, and having determinant zero, which is a *non*-equation.

To deal with this, let

$$GL'(n,k) \;:=\; \{(g,d,d') : G \in M(n,k),\; d,d' \in k,\; \det g = d,\; dd' = 1\} \;\subseteq\; M(n,k) \oplus k^2.$$

This is an algebraic set, as we are picking it out using equations, but it is in one-to-one correspondence with $GL(n,k)$. Moreover, $GL'(n,k)$ becomes a group in AlgSet_k with multiplication defined as

$$(g_1,d_1,d_1')(g_2,d_2,d_2') \;:=\; (g_1g_2, d_1d_2, d_1'd_2'),$$

and inverses defined as

$$(g,d,d')^{-1} \;:=\; (g^{-1}, d', d),$$

where g^{-1} can be written using the minors of g, and d' as a substitute for $(\det(g))^{-1}$, which is therefore a polynomial.

Thus, G becomes a group in AlgSet_k via

$$G' \;:=\; \{(g,d,d') : g \in GL'(n,k) : P_1(g) = \cdots = P_n(g) = 0\} \;\subseteq\; M(n,k) \oplus k^2.$$

The converse is harder; see Milne's Algebraic Groups, section 4d. ∎