# LINEAR ALGEBRAIC GROUPS: LECTURE 11

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## 1. Recap

We started this course by looking at Klein geometry, where highly symmetric geometries could be described via subgroups of $GL(n,k)$. This approach reflects the thinking in the mathematics of the 1800's.

More recently, we looked at the "modern approach", where any algebraic set can be viewed intrinsically through its commutative algebra of regular functions.

Finally, we progress to schemes, and in particular affine schemes, ideas developed largely through the work of Grothendieck.

Now, we want to look at Klein geometry with algebraic groups. After considering how much material we *could* cover in Milne's books, we will choose, as a goal, to look at "**The Big Theorem**".

## 2. Borel Subgroups and Bruhat Cells

As a warm-up, suppose $G$ is a linear algebraic group over an algebraically closed field $k$ (Note: For general fields, the statements that follow become more complicated).

**Definition.** A Borel subgroup $B \subseteq G$ is a connected, solvable subgroup that's maximal (not properly contained in any other connected, solvable subgroup).

Here, connected is defined using the Zariski topology on $G$, a topology defined on any algebraic set (or any affine variety) by declaring the sets picked out by some polynomial equations to be closed.

**Example.** Consider the plane in $\mathbb{C}^2$ defined as

$$S_1 \;=\; \left\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\right\}.$$

This is closed in the Zarisky topology. Moreover, $S_2 \subseteq S_1$ defined by

$$S_2 \;=\; \left\{(z_1, z_2) \in \mathbb{C}^2 : z_2(z_1^7 - 1) = 0\right\}$$

is a closed in the Zarisky topology, and represents the seventh roots of unity in the $z_1$ plane.

Also, a subgroup $H \subseteq G$ is solvable if there exist normal subgroups

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$$

such that each quotient $H_i/H_{i-1}$ is abelian.

**(Key) Example.** If $G = GL(n)$ then the subgroup $H$ of upper triangular matrices is a Borel subgroup. This subgroup preserves the complete flags $F_n$, which we defined previously as

$$F_n \;:=\; V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = k^n,$$

where each $V_i$ is a vector subspace of $k^n$ of dimension $i$. This is fundamental to the Klein approach, where we associated geometry to algebra by considering groups that map particular types of figures to figures of the same type (such as complete flags).

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Now, our new definition generalizes Borel subgroups to any linear algebraic group $G$. Moreover, our Big Theorem about Borel subgroups will only apply when $G$ is nice, in a particular way: It should be reductive. Suppose our field $k$ has characteristic zero, which means that

$$\underbrace{1+1+\cdots+1}_{n-\text{times}} \neq 0$$

for all $n \in \mathbb{N}$. $\mathbb{C}$ has characteristic zero, while both $\mathbb{F}_q$ and $\overline{\mathbb{F}}_q$ have characteristic $p$ (where $q = p^n$), even though $\overline{\mathbb{F}}_q$ is algebraically closed.

**Definition.** A group $G$ is reductive if it is connected, and any finite representation of $G$ is a direct sum of irreducible representations.

Even though we aren't here to talk about representation theory, a representation of a group $G$ is a homomorphism $\rho : G \to GL(N)$ , where $N$ need not equal $n$, of affine algebraic groups in the category $\mathrm{AffAlg}_k$. We say that $\rho = \rho' \oplus \rho''$ if, possibly after a change of basis on $k^N$,

$$\rho(g) \;=\; \left(\begin{array}{c|c} \rho'(g) & 0 \\ \hline 0 & \rho''(g) \end{array}\right).$$

A nonzero representation $\rho$ is irreducible if for any vector space $V \subseteq k^N$,

$$\rho(g) : V \to V$$

only if $V = \{0\}$, i.e. no nontrivial subspace smaller than $k^N$ is fixed by $\rho(g)$.

**Examples.** These linear algebraic groups are reductive:

$$GL(n,\mathbb{C}),\ Sl(n,\mathbb{C}),\ O(n,\mathbb{C}),\ SO(n,\mathbb{C}),\ Sp(n,\mathbb{C}),$$

where $Sp(n,\mathbb{C})$ is the collection of symplectic transformations, or symplectomorphisms, on $\mathbb{C}^n$. These are some of the "classical" groups.

**Theorem** (Big Theorem, intial)**.** *Suppose $G$ is a reductive linear algebraic group over an algebraically closed field $k$. Then there exists a Borel subgroup $B$ of $G$, and any two Borel subgroups $B$ and $B'$ of $G$ are conjugate (so there exists a $g \in G$ such that $B' = gBg^{-1}$).*

We may frequently talk about "the" Borel subgroup of $G$, as they are all conjugate. A point in the set $G/B$ (as $B$ need not be normal) is called a complete flag. The set $G/B$ is a disjoint union of Bruhat cells, each in bijection with $k^i$ for some $i$.

There's a Bruhat cell for each element in the Weyl group $W$, a finite group associated to $G$. Each Bruhat cell gives a $G$-invariant relation on complete flags

$$R \;\subseteq G/B \times G/B,$$

where by $G$-invariance

$$(f,f') \in R \;\Leftrightarrow\; (gf,gf') \in R \qquad \text{for all } (f,f') \in R.$$

Indeed, we get every $G$-invariant relation from the Bruhat type, using "or" or "$\cup$". If $R, R'$ are relations, then so is $R \cup R'$. So we get a complete picture of the interesting types of figures of the Klein geometry associated to $G$, and their $G$-invariant relations, once we understand this theorem.

**Example.** Consider $GL(n,k)$, where we can take

$$B \;=\; \{\text{upper triangular matrices in } GL(n,k)\}$$

to be the Borel subgroup. The Weyl group is then $W = S_n$, the permutation group on $n$ elements. The subgroup $B$ preserves the canonical complete flag

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = k^n,$$

where we have $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with a one in the $i^{\text{th}}$ coordinate, and $V_1 = \langle e_1 \rangle$, $V_2 = \langle e_1, e_2 \rangle$ etc. This implies that $g \in B$ if and only if $ge_i$ is a linear combination of $e_1, e_2, \ldots, e_i$, matching that $B$ consists of upper triangular matrices.

But $G/B$ is the collection of <u>all</u> complete flags. What are the other complete flags, and what Bruhat cells do they live in?

Consider $n = 3$, so $W = S_3$, which is generated by

$$s_1 = \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix}.$$

We can write the Weyl group as

$$\begin{tikzcd}
 & \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
\begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \arrow[d, "s_2"'] & & \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \arrow[d, "s_1"] \\
\begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \arrow[dr, "s_1"'] & & \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \arrow[dl, "s_2"] \\
 & \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} &
\end{tikzcd}$$

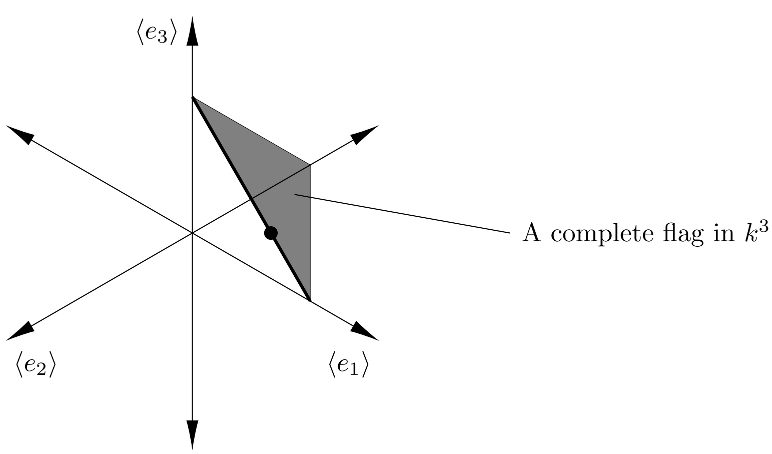
Notice that $s_1 s_2 s_1 = s_2 s_1 s_2$. Our favorite flag is

$$V_1 = \langle e_1 \rangle, \qquad V_2 = \langle e_1, e_2 \rangle, \qquad V_3 = \langle e_1, e_2, e_3 \rangle = k^3.$$

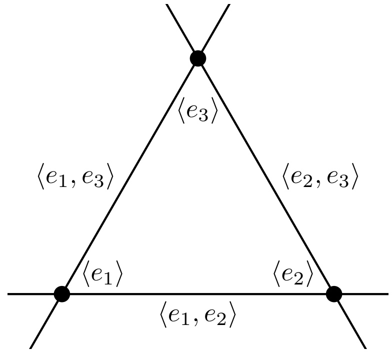
Given any $\sigma \in S_3$, we get a new flag

$$V_1' = \langle \sigma e_1 \rangle, \qquad V_2 = \langle \sigma e_1, \sigma e_2 \rangle, \qquad V_3 = \langle \sigma e_1, \sigma e_2, \sigma e_3 \rangle = k^3.$$

Since $|S_3| = 6$, there are six complete flags, one in each Bruhat cell! But what do they look like? In $k^3$, we can picture it as



However, drawing in 3 dimensions is mildly annoying, so let's do it in a projective sense. We can draw our 1- and 2-dimensional subspaces as points and lines instead:




Here, the dots represent the one dimensional subpaces (points in $kP^2$) generated by the various $e_i$, while the lines are projective lines generated by pairs $e_i$ and $e_j$ with $j \neq i$. Notice that the permutation $\begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}$ gives us the complete flag with

$$p_1 = \langle e_1 \rangle\,, \qquad \ell_1 = \langle e_1, e_2 \rangle\,.$$

Similarly, the permutation $\begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix}$ gives us the complete flag with

$$p_2 = \langle e_2 \rangle\,, \qquad \ell_2 = \langle e_1, e_2 \rangle\,,$$

which is a different point on the same line. Flags of this type - different points on the same line - form a Bruhat cell of dimension one, as we can choose any point on the line. Here's a comparison between the Weyl group $S_3$ and the associated complete flags/Bruhat cells:

$$\begin{tikzcd}
 & \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
\begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \arrow[d, "s_2"'] & & \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \arrow[d, "s_1"] \\
\begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \arrow[dr, "s_1"'] & & \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \arrow[dl, "s_2"] \\
 & \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} &
\end{tikzcd}$$