LINEAR ALGEBRAIC GROUPS: LECTURE 12

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1. Recap from Lecture 11

Given our "favorite" complete flag in k^n,

$$F = V_1 \subset V_2 \subset \cdots \subset V_n = k^n,$$

we can choose a (standard) basis $\{e^i\}_{i=1}^n$ such that

$$V_i = \langle e_1, e_2, \ldots, e_i \rangle$$

for all i. The Weyl group of permutations, S_n, then generate the rest of the complete flags. Each $\sigma \in S_n$ generates a unique flag generated by

$$V_i' = \langle \sigma e_1, \sigma e_2, \ldots, \sigma e_i \rangle .$$

The Big Theorem says that each new flag lies in a unique Bruhat cell in the (complete) flag variety: the set of complete flags, which is isomorphic to $GL(n,k)/B$, where B is the Borel subgroup that fixes our favorite flag.

2. Which Bruhat Cell?

In order to identify the unique Bruhat cell for each flag, we need to find a convenient way to represent our flags. Let's start in k^3, with our favorite flag being represented as a matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} \leftarrow & e_1 \\ \leftarrow & e_2 \\ \leftarrow & e_3. \end{matrix}$$

Notice that V_1 is represented as $(1\,0\,0)$ which means the span of e_1, while V_2 is represented as the first two rows, which means the span of both e_1 and e_2. Finally, k^3 is generated by all three rows, which in this case is e_1, e_2 and e_3. In the language of projective geometry let's declare V_1 to be p, a point in kP^2, and V_2 to be ℓ, a line in kP^2.

Of course, any matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \begin{matrix} \leftarrow & v_1 \\ \leftarrow & v_2 \\ \leftarrow & v_3, \end{matrix}$$

where $\lambda_i \in k\backslash\{0\}$ would describe the same subspaces (and therefore, complete flag), but we are really only concerned with the flag - not its generators.

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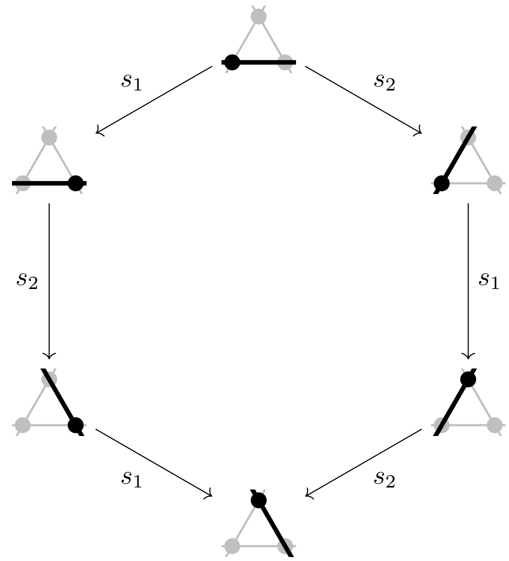
Using incidence relations and matrices, we want to fill in our hexagonal representation of the Weyl group:

$$\begin{tikzcd}
 & \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
\begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \arrow[d, "s_2"'] & & \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \arrow[d, "s_1"] \\
\begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \arrow[dr, "s_1"'] & & \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \arrow[dl, "s_2"] \\
 & \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} &
\end{tikzcd}$$



in a way that describes both the unique Bruhat cells, and the associated types of complete flags.

At each vertex of the hexagon we will call the one-dimensional subspace p' and the two-dimensional subspace ℓ'. They give a flag $p' \subset \ell$. We get this from the element of S_3 indicated at this vertex of the hexagaon. Each flag has its own relation to our favorite flag $p \in \ell$. Let us examine some of these.

For the top vertex, it should be clear that we have $p' = p$, and $\ell' = \ell$. What about the upper left vertex, created by s_1(transposing 1 and 2) acting on our original flag This flag satisfies $\ell' = \ell$ and $p' \subset \ell$, *but no more*, where by no more we mean it does *not* satisfy the relations of the top vertex. We will use this convention to write all the incidence relations, so lower vertices *do not* satisfy some (or all) of the incidence relations of higher vertices.

What do these relations really describe? Not just our representative flag, rather the collection of all flags that satisfy $\ell' = \ell$ and $p' \subset \ell$, but $p' \neq p$. In terms of generators, we can satisfy $p' \neq p$ by having a generator for $V_1' = p'$ which is independent of that for $V_1 = p$, say

$$p' = \langle (1\ 1\ 1) \rangle .$$

However, this wouldn't work, as we also require that $p' \subset \ell$, so there can be no e_3 component in a generator of p'. We can also normalize relative to the e_2 component, and say that any subspace of the form

$$p' = \langle (* \; 1 \; 0) \rangle$$

will satisfy $p' \neq p$, but $p' \subset \ell$. This does work, as any such vector is in ℓ, and is linearly independent of (1 0 0).

To build $V_2' = \ell'$, we then need to choose a vector that makes $\ell' = \ell$, and there is a natural choice - the vector (1 0 0). Finally, any third vector will complete k^3, so can simply choose (0 0 1). In matrix form, this gives us

$$\begin{pmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{cc} \leftarrow & v_1 \\ \leftarrow & v_2 \\ \leftarrow & v_3. \end{array}$$

Notice that this has one degree of freedom, in the sense that $*$ is a wildcard that can be any element in k.

Continuing, if we look at the lower left vertex we have (pictorially) $p' \subset \ell$ but $\ell' \neq \ell$. As in our previous example, we have know that our p' can be generated by something of the form $(* \; 1 \; 0)$. However, to satisfy $\ell' \neq \ell$, we must choose a generator with some e_3 component, so after normalizing we can consider the generator $(* \; 0 \; 1)$. In order to assure that the three generators are all linearly independent, we can choose (1 0 0) as our final generator, so the matrix in the lower left is

$$\begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{array}{cc} \leftarrow & v_1 \\ \leftarrow & v_2 \\ \leftarrow & v_3. \end{array}$$

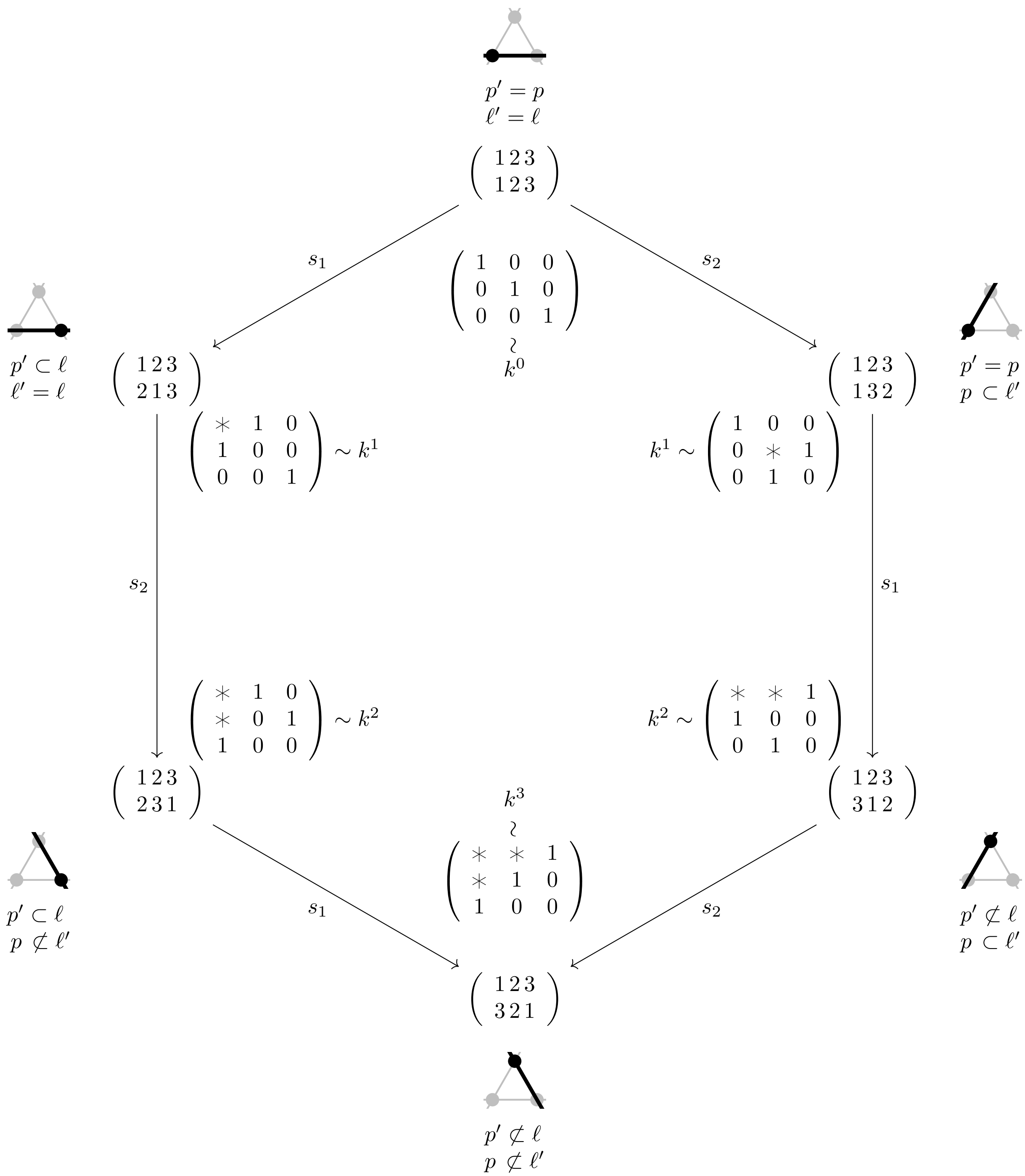
Here, we have two wildcards, which can be any element in k. This places such a flag in a Bruhat cell of the form k^2.

Finally, lets look at the lowest vertex of our hexagon. In this case, we have $p' \not\subset \ell$ and $p \not\subset \ell'$. To satisfy $p' \not\subset \ell$, we can choose as our first generator anything of the form $(* \; * \; 1)$, so it will always have an e_3 component. To satisfy $p \not\subset \ell'$, we can choose something of the form $(* \; 1 \; 0)$, so it always contains an e_2 component. Again, adding a vector of the form (1 0 0) will then generate all of k^3. This means we have a matrix

$$\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{array}{cc} \leftarrow & v_1 \\ \leftarrow & v_2 \\ \leftarrow & v_3. \end{array}$$

Such a matrix (and flag) will live in a Bruhat cell of the form k^3, as shown by the three wildcards. In a similar manner, we can construct the associated incidence relations and matrices for the final two vertices.

In a massive batch of correlations, we have



3. Reading the Results

Through the use of wildcards ($*$), we can see that the upper vertex represents a single point, $1 = k^0$, while the next highest are a pair of k^1. Continuing down the entire hexagon, we get that

$$Gl(3)/B \;\cong\; 1 + k + k + k^2 + k^2 + k^3.$$

Over a finite field $k = \mathbb{F}_q$, the number of complete flags in k^3 is

$$|Gl(3)/B| \;=\; 1 + q + q + q^2 + q^2 + q^3,$$

which is $[\,3\,]_q!$. Naturally, in n-dimensions, we have $|GL(n)| = [\,n\,]_q!$, a sum over the $n!$ Bruhat cells (the cardinality of the permutation group) of $Gl(n)/B$, where a cell of dimension i contributes a q^i to the sum.

Our hexagon is a 2-dimensional polytope. In the next lecture, we will look at S_4 as the Weyl group, and we will require working in three dimensions.