LINEAR ALGEBRAIC GROUPS: LECTURE 13

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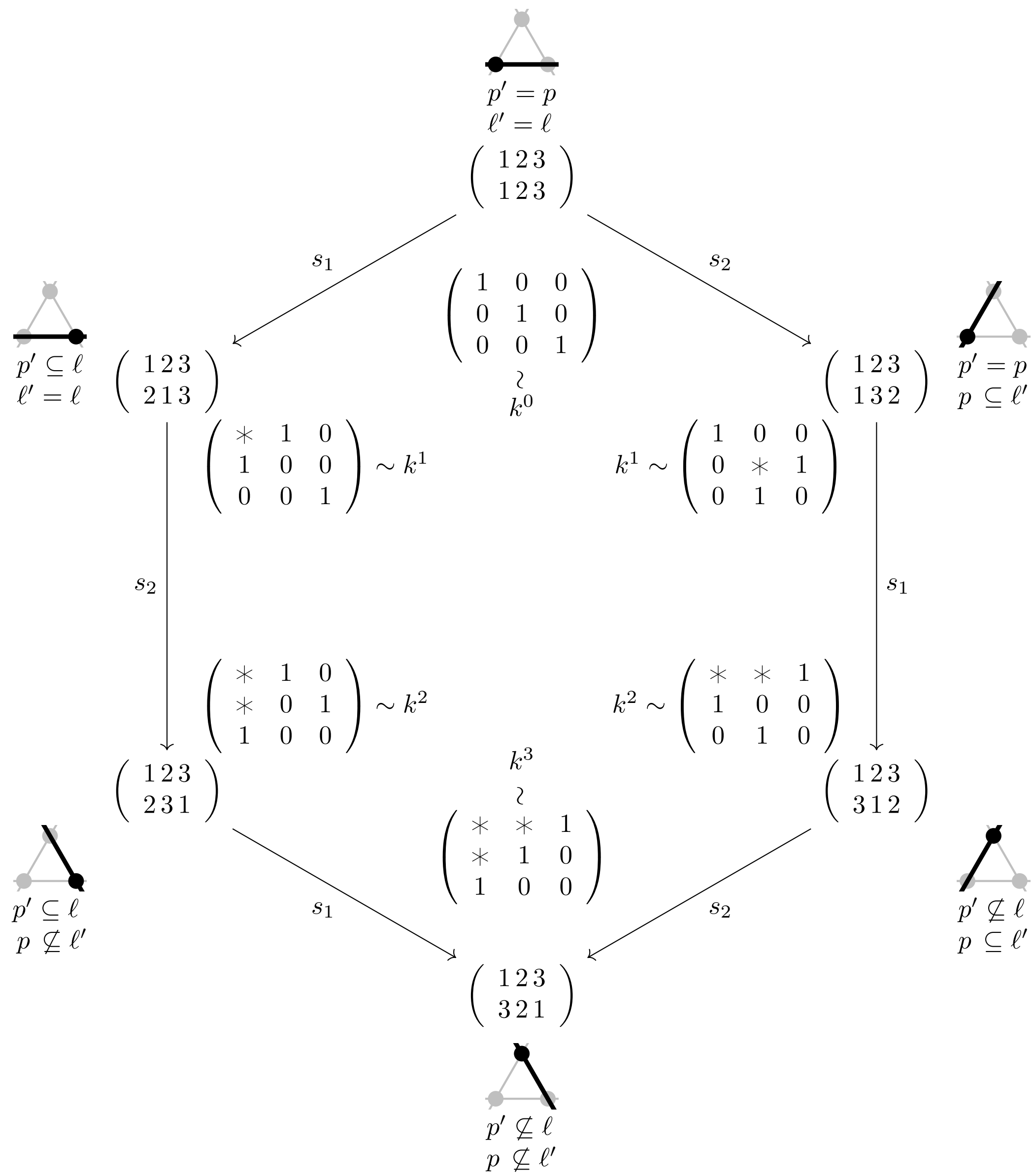
1. Recap from Lecture 12

In the last lecture, we looked at the group $GL(3)$ and its Weyl group S_3. We constructed a hexagon that represented the Bruhat cells, showing the correspondence between incidence relations and elements in the Weyl group:



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We can use this as a model to proceed to arbitrary $GL(n)$, and will begin with $GL(4)$.

2. Permutahedrons in Higher Dimensions

If $\{e_1, e_2, \dots, e_n\}$ form a basis for k^n, we have our "favorite" complete flag

$$F \;=\; \{0\} \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, e_2, \dots, e_n \rangle \;=\; k^n.$$

For $GL(n, l) = GL(n)$, where the Weyl group is $W = S_n$, for any $\sigma \in S_n$ we get a new flag

$$F_\sigma \;=\; \{0\} \subseteq \langle \sigma(e_1) \rangle \subseteq \langle \sigma(e_1), \sigma(e_2) \rangle \subseteq \cdots \subseteq \langle \sigma(e_1), \sigma(e_2), \dots, \sigma(e_n) \rangle \;=\; k^n.$$

Each $\sigma \in S_n$ determines a Bruhat cell C_σ, which is a subset of the flag variety $GL(n)/B$. C_σ contains F_σ, but also other flags having the same relation to F as F_σ. Writing a flag F' as

$$F' \;=\; \{0\} \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, v_2, \dots, v_n \rangle \;=\; k^n,$$

where $\{v_i\}$ is a basis for k^n, we get a matrix

$$\begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & & \\ \vdots & & \ddots & \vdots \\ v_{n1} & \cdots & & v_{nn} \end{pmatrix}$$

whose i^{th} row is associated to the components of v_i. We can then use row operations to make as many entries zero as possible without changing the flag F'. All the matrices corresponding to some $F' \in C_\sigma$ will then look like

$$\begin{pmatrix} e_{\sigma(1)1} & e_{\sigma(1)2} & e_{\sigma(1)n} \\ \vdots & \ddots & \vdots \\ e_{\sigma(n)1} & \cdots & e_{\sigma(n)n} \end{pmatrix}$$

with nonzero entries only in particular relations, which we showed as stars.

Now, S_n has a presentation with <u>elementary transpositions</u> as generators (for $1 \le i \le n-1$):

$$s_i \;=\; \begin{array}{ccccccc} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ | & | & & \multicolumn{2}{c}{\times} & & | \\ 1 & 2 & \cdots & i+1 & i & \cdots & n \end{array}$$

and relations:

$$s_i^2 = 1; \qquad s_i s_{i+1} s_1 = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \text{ if } |i-j| > 1.$$

Any permutation $\sigma \in S_n$ has a length $\ell(\sigma)$, which si the minimum number of s_i's needed to write σ as a product of s_i's.

Theorem. *$C_\sigma \cong k^{\ell(\sigma)}$ as sets.*

Proof. (Idea) The flag $F_{\sigma s_i}$ is obtained by changing the i-dimensional space in the flag F_σ. More generally, any flag in $C_{\sigma s_i}$ can be obtained by changing the i-dimensional vectore space in some flag in C_σ, giving one extra degree of freedom, so a flag in C_σ will be determined by $\ell(\sigma)$ elements of k (the $\ast$'s in the chart). $\square$

In fact, $GL(n)/B$ is a disjoint union of the Bruhat cells, so

Corollary. *For any prime power q,*

$$[n]_q! \;=\; \sum_{\sigma \in S_n} q^{\ell(\sigma)}.$$

Proof. A while ago we showed that if $k = \mathbb{F}_q$, then

$$|GL(n)/B| \;=\; [n]_q!$$

However, we now know that

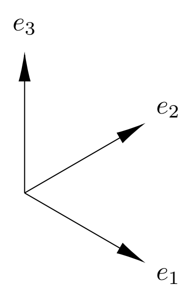
$$GL(n)/B \;=\; \bigsqcup_{\sigma \in S_n} C_\sigma,$$

so

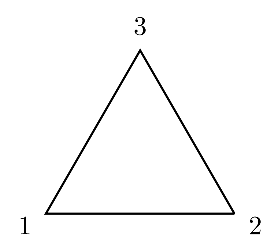
$$[n]_q! \;=\; |GL(n)/B| \;=\sum_{\sigma \in S_n} |C_\sigma| \;=\; \sum_{\sigma \in S_n} q^{\ell(\sigma)}.$$

□

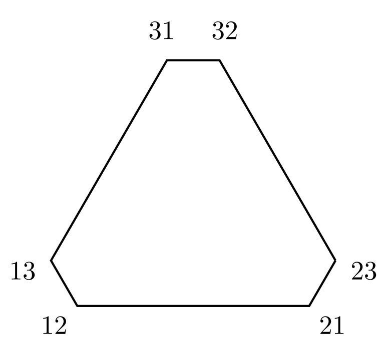
The elements of S_n are the vertices of a polytope called a permutahedron, which for $n = 3$ is our hexagon. You get it by taking the $n-1$ simplex (triangle, tetrahedron etc.) and "omnitruncating" it. For $n = 3$,



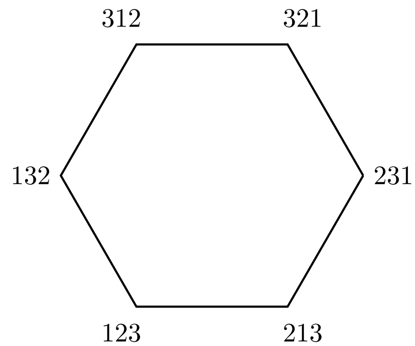
gives



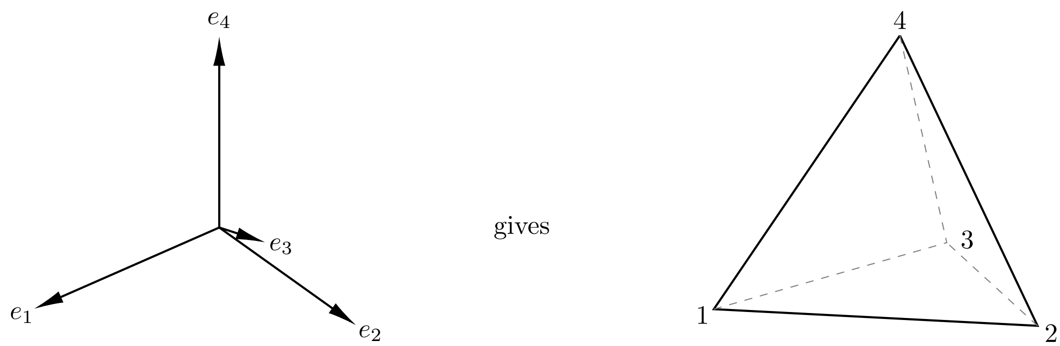
Omnitruncating the corners then gives us a hexagon:



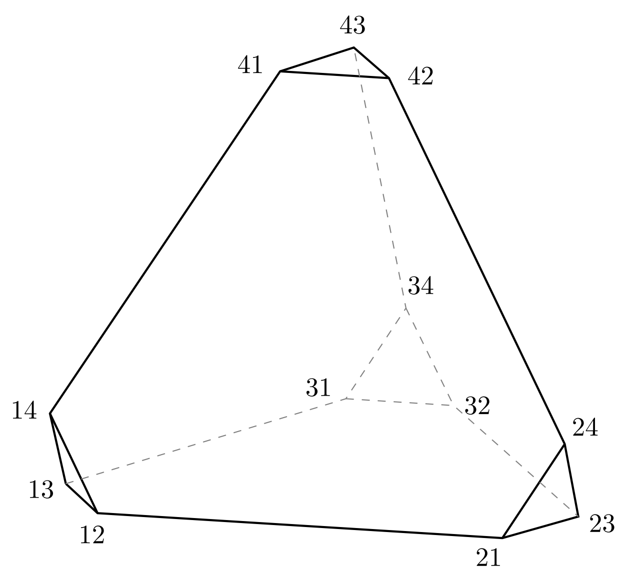
or more artistically,



For $n = 4$, we begin with axes, and build a tetrahedron, where the vertices represent point-type flags in kP^3 :

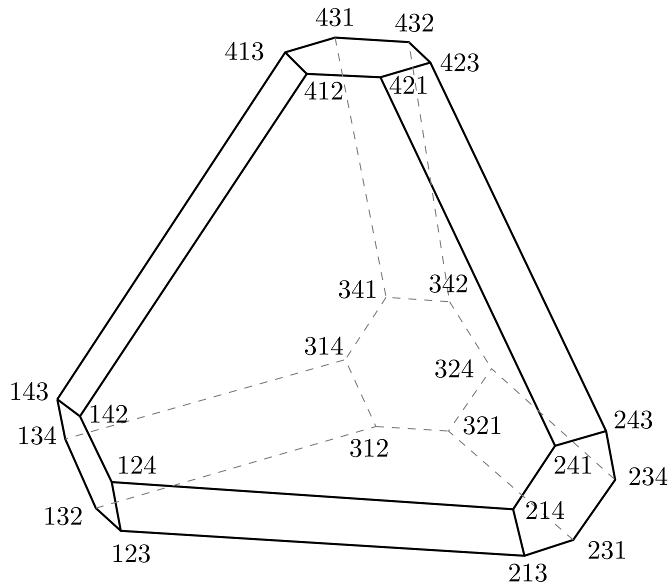


We can think of the vertices represents as $P(4,1)$ - the possible arrangements of 4 objects taken one at a time - corresponding to a one dimensional subspace from our 4 basis vectors. When we first truncate, we then have vertices that represent flags of the point-line type in kP^3 :

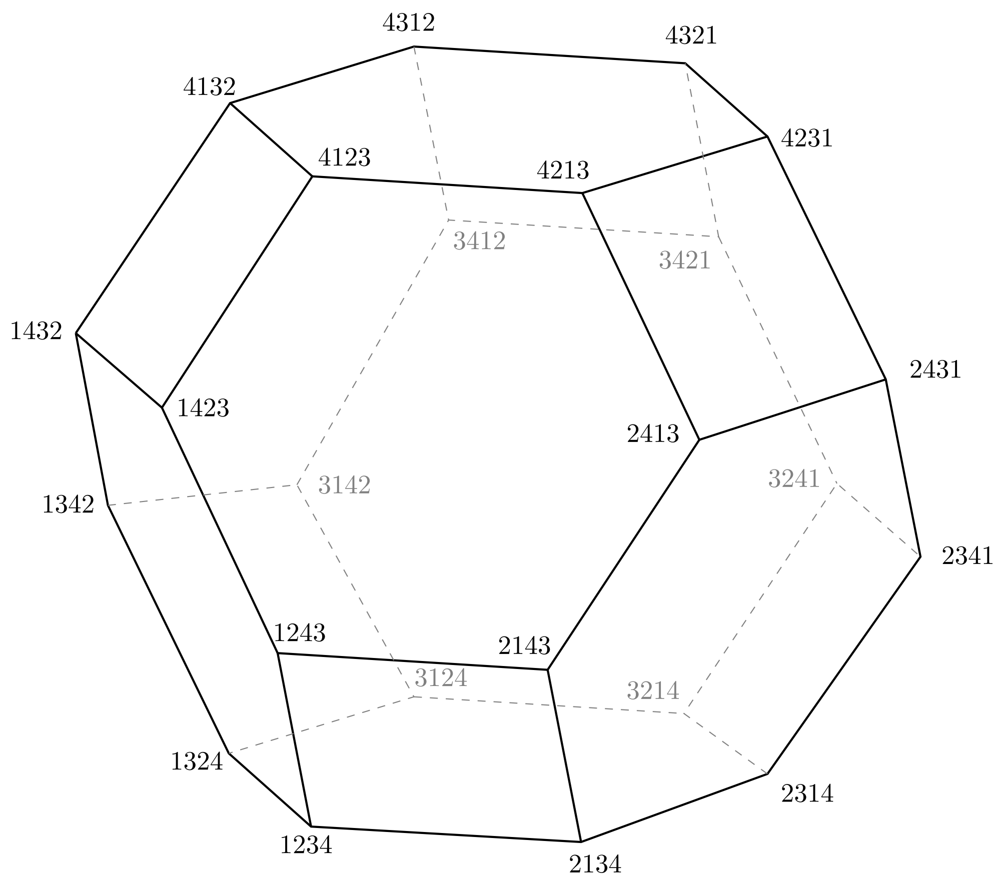


Notice that the vertices also can be thought of as $P(4,2)$, the permutation of 4 objects taken two at a time.

When we do the final truncation, we arrive at complete flags, which are point-line-plane type in kP^3 :



As written, the vertices are now $P(4,3)$. Finally, we can again make it more "artistic", which is to say regular. This is the permutahedron for $GL(4)$, with all sides of the same length:



By writing the ordered index of all four basis elements for our flag, it is now clear that each edge is an elementary transposition. This shows that each F_σ lies a minimum of $\ell(\sigma)$ edges to move back to our favorite flag, represented here as 1234.